

Maxwell's equations in media as a contact Hamiltonian vector field and its information geometry

– An approach with a bundle whose fiber is a contact manifold –

Shin-itiro Goto

Department of Applied Mathematics and Physics,
Graduate School of Informatics, Kyoto University,
Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan

February 21, 2017

Abstract

It is shown that Maxwell's equations in media without source can be written as a contact Hamiltonian vector field restricted to a Legendre submanifold, where this submanifold is in a fiber space of a bundle and is generated by either electromagnetic energy functional or co-energy functional. Then, it turns out that Legendre duality for this system gives the induction oriented formulation of Maxwell's equations and field intensity oriented one. Also, information geometry of the Maxwell fields is introduced and discussed.

1 Introduction

Contact geometry is often referred to as an odd-dimensional twin of symplectic geometry and then it has been studied from purely mathematical viewpoints[1]. On the other hand, there are several applications in science and foundation of engineering. These applications include equilibrium thermodynamics[2, 3, 4, 5], nonequilibrium thermodynamics[6, 7], statistical mechanics[8, 9, 10], fluid mechanics[11], control theory[12], statistical theory for non-conservative system[13], electric circuits[14], dissipative mechanical systems[15], and so on. In general, if geometric theories of mathematical disciplines are ascribed to the same geometry, then it can be expected that there are links among these disciplines. These links may give a unified geometric picture of such disciplines. Such an example is found in contact geometry, where information geometry is linked to contact geometric thermodynamics[7, 8]. Here information geometry is a geometrization of parametric statistics [16], and it has been applied to various disciplines including equilibrium statistical mechanics[17], control theory[18], and so on. In Ref.[19], several simple electric circuit models without any external source have been discussed in terms of contact and information geometries. Since electromagnetic media can be seen as distributed element models of electrical circuits, one can expect that Maxwell's equations without source in media can be written as some kind of contact geometry and information geometry.

There exists a history of developing geometric formulations of Maxwell's equations for describing electromagnetic fields. The most well-known one is that these equations are described on a 4-dimensional pseudo Riemannian manifold[20, 21, 22]. Also, it has been shown that Maxwell's vacuum equations are written as an infinite dimensional Hamiltonian systems[23]. Furthermore, since Dirac structures on manifolds are known to be a geometric generalization of phase spaces of Hamiltonian systems[24, 25], one expects that Maxwell's equations can be written with an extension of a Dirac structure. It then has been shown that Maxwell's equations are described as a Hamiltonian system with respect to a Stokes-Dirac structure[26]. Note that some wave solutions to Maxwell's equations can be described in terms of contact geometry[29], and some electric circuits are described as a Dirac structure[27, 28].

In this paper it is shown how Maxwell's equations without source in media are described in terms of a bundle whose fiber space is a contact manifold. To this end, an idea of such a bundle and how a contact Hamiltonian vector field will be formulated on such a bundle are discussed. In this formulation, an analogue of convexity of the electromagnetic energy functional is emphasized, and then one notices that the property of such functionals allows us to use convex analysis so that Legendre duality should be explored in this context. It will then be shown that this duality leads to an induction field oriented formulation of Maxwell's equations and field intensity oriented formulation. Also, it will turn out that the existence of such convex functional leads to an analogue of a dually flat space introduced in information geometry. These formulations can shed light on how Legendre transform and convex functions play a role in electromagnetism.

2 Mathematical preliminaries

In this section mathematical symbols and tools are fixed and these will be used in the following sections. Mathematical objects in this paper are assumed smooth and real. From a viewpoint of dynamical systems theory, Maxwell's equations are an infinite dimensional system. To formulate these equations the dimension of phase space should be infinite. To this end, one starts with the finite dimensional case.

2.1 Contact geometry

The definition of contact manifold below follows the one often used in contact geometric equilibrium thermodynamics (see Ref. [3]).

Definition 2.1. (*Contact manifold*): Let \mathcal{C} be a $(2n+1)$ -dimensional manifold ($n = 1, 2, \dots$). If \mathcal{C} carries a 1-form λ such that

$$\lambda \wedge \underbrace{d\lambda \cdots d\lambda}_n \neq 0,$$

then the pair (\mathcal{C}, λ) is referred to as a $((2n+1)$ -dimensional) contact manifold.

In this paper the following coordinate system is often used.

Theorem 2.1. (*Darboux's theorem*): Let (\mathcal{C}, λ) be a contact manifold. Then there exists the local coordinate system (x, p, z) such that $\lambda = dz - p_a dx^a$ with $x = \{x^1, \dots, x^n\}$ and $p = \{p_1, \dots, p_n\}$.

In this paper, Einstein notation, when allowed index variables appear twice in a single term it implies that all the values of the index are summed, is used. Note that, instead of λ in Theorem 2.1, another convention exists in the literature.

Definition 2.2. (*Darboux coordinates*): The coordinate system stated in Theorem 2.1 is referred to as the Darboux coordinates or canonical coordinates.

The following transformation group preserves the contact structure $\ker(\lambda) := \{X \in \Gamma T\mathcal{C} | \iota_X \lambda = 0\}$ for a given contact manifold (\mathcal{C}, λ) .

Definition 2.3. (*Contact transformation group*): Let (\mathcal{C}, λ) be a contact manifold and $\{\Phi_\varsigma\} : \mathcal{C} \rightarrow \mathcal{C}$ a set of elements of a group. If Φ_ς is such that

$$\Phi_\varsigma^* \lambda = f_\varsigma \lambda$$

where f_ς is some non-vanishing function and $\Phi_\varsigma^* \lambda$ is a pull-back of λ , then $\{\Phi_\varsigma\}$ is referred to as a contact transformation group.

In the following sections the phase spaces of Maxwell's equations for media will be described in terms of a bundle whose fiber space is a contact manifold. In particular, the phase space of such equations is described on a Legendre submanifold of the fiber space. Roughly speaking these objects are appropriate for describing dynamical systems as partial differential equations, in addition that the standard contact geometry is appropriate for describing dynamical systems written as ordinary differential equations.

To describe dynamical systems written as ordinary differential equations on contact manifolds, one needs the following.

Definition 2.4. (*Reeb vector field*): Let (\mathcal{C}, λ) be a contact manifold, and \mathcal{R} a vector field on \mathcal{C} . If \mathcal{R} satisfies

$$\iota_{\mathcal{R}}\lambda = 1, \quad \text{and} \quad \iota_{\mathcal{R}}d\lambda = 0,$$

then \mathcal{R} is referred to as the Reeb vector field or the characteristic vector field.

Here ι_X is the interior product associated with a vector field X . It is known that the Reeb vector field is uniquely determined and that the Darboux coordinate expression of \mathcal{R} is $\partial/\partial z$.

Roughly speaking, the following vector fields are obtained by the infinitesimal transforms of contact transforms for a given contact manifold (\mathcal{C}, λ) .

Definition 2.5. (*Contact vector field*): Let (\mathcal{C}, λ) be a contact manifold and X a vector field on \mathcal{C} . If X satisfies $\mathcal{L}_X\lambda = f\lambda$ with some function f , then X is referred to as a contact vector field.

Here \mathcal{L}_X denotes the Lie derivative with respect to a vector field X . To realize a contact vector field, one can take a contact Hamiltonian vector field. With the Reeb vector field, one defines the following.

Definition 2.6. (*Contact Hamiltonian vector field*): Let (\mathcal{C}, λ) be a contact manifold, h a function on \mathcal{C} , and X_h a vector field on \mathcal{C} . If X_h satisfies

$$\iota_{X_h}\lambda = h, \quad \text{and} \quad \iota_{X_h}d\lambda = -(dh - (\mathcal{R}h)\lambda), \quad (1)$$

then X_h is referred to as the contact Hamiltonian vector field associated with h , and h a contact Hamiltonian.

Remark 2.1. Applying the Cartan formula $\mathcal{L}_X\alpha = d\iota_X\alpha + \iota_Xd\alpha$ for a q -form field α , one has a basic property for a contact Hamiltonian vector field :

$$\mathcal{L}_{X_h}\lambda = (\mathcal{R}h)\lambda.$$

Contact Hamiltonian vector fields are to be used for describing various equations on contact manifolds in the following sections.

By straightforward calculations one can show the following.

Proposition 2.1. (*Coordinate expression of contact Hamiltonian vector field*): The Darboux coordinate expression of (1) is given as

$$X_h = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{p}_a \frac{\partial}{\partial p_a} + \dot{z} \frac{\partial}{\partial z},$$

where

$$\dot{x}^a = -\frac{\partial h}{\partial p_a}, \quad \dot{p}_a = \frac{\partial h}{\partial x^a} + p_a \frac{\partial h}{\partial z}, \quad \dot{z} = h - p_a \frac{\partial h}{\partial p_a}. \quad (2)$$

Let ϕ_h be an integral curve of X_h such that $\phi_h : \mathbb{T} \rightarrow \mathcal{C}, (t \mapsto (x, p, z))$ is a map with some $\mathbb{T} \subseteq \mathbb{R}$. Then $\dot{}$ denotes the derivative with respect to t . In this case, one has dynamical systems. Physically this $t \in \mathbb{T}$ is interpreted as time. In what follows contact vector fields are always treated as dynamical systems and integral curves are focused when a contact Hamiltonian vector field is given.

Remark 2.2. Unlike the case of autonomous Hamiltonian vector fields, contact Hamiltonian vector fields need not be conserved, since $\mathcal{L}_{X_h}h = (\mathcal{R}h)h$ does not vanish in general.

In applications of contact geometry, Legendre submanifolds play various roles. The definition of Legendre submanifold is as follows.

Theorem 2.2. (*Maximal dimension of integral submanifold*): Let (\mathcal{C}, λ) be a $(2n+1)$ -dimensional contact manifold. The maximal dimension of integral submanifold of $\lambda = 0$ is n .

Definition 2.7. (*Legendre submanifold*): Let (C, λ) be a contact manifold and \mathcal{A} a submanifold of C . If \mathcal{A} is a maximal dimensional integral submanifold of λ , then \mathcal{A} is referred to as a Legendre submanifold.

It has been known that local expressions of Legendre submanifolds are described by some functions on contact manifolds.

Theorem 2.3. (*Local expressions of Legendre submanifolds, [1]*): Let (C, λ) be a $(2n + 1)$ -dimensional contact manifold, and (x, p, z) the canonical coordinates such that $\lambda = dz - p_a dx^a$ with $x = \{x^1, \dots, x^n\}$ and $p = \{p_1, \dots, p_n\}$. For any partition $I \cup J$ of the set of indices $\{1, \dots, n\}$ into two disjoint subsets I and J , and for a function $\phi(x^J, p_I)$ of n variables $p_i, i \in I$, and $x^j, j \in J$ the $(n + 1)$ equations

$$x^i = -\frac{\partial \phi}{\partial p_i}, \quad p_j = \frac{\partial \phi}{\partial x^j}, \quad z = \phi - p_i \frac{\partial \phi}{\partial p_i}, \quad (3)$$

define a Legendre submanifold. Conversely, every Legendre submanifold of (C, λ) in a neighborhood of any point is defined by these equations for at least one of the 2^n possible choices of the subset I .

Definition 2.8. (*Legendre submanifold generated by function*): The function ϕ used in Theorem 2.3 is referred to as a generating function of the Legendre submanifold. If a Legendre submanifold \mathcal{A} is expressed as (3), then \mathcal{A} is referred to as a Legendre submanifold generated by ϕ .

The following are examples of local expressions for Legendre submanifolds.

Example 2.1. Let (C, λ) be a $(2n + 1)$ -dimensional contact manifold, (x, p, z) the canonical coordinates such that $\lambda = dz - p_a dx^a$ with $x = \{x^1, \dots, x^n\}$ and $p = \{p_1, \dots, p_n\}$, and ψ a function of x only. Then, the Legendre submanifold \mathcal{A}_ψ generated by ψ with $\Phi_{C\mathcal{A}_\psi} : \mathcal{A}_\psi \rightarrow C$ being the embedding is such that

$$\Phi_{C\mathcal{A}_\psi} \mathcal{A}_\psi = \left\{ (x, p, z) \in C \mid p_j = \frac{\partial \psi}{\partial x^j}, \text{ and } z = \psi(x), \quad j \in \{1, \dots, n\} \right\}. \quad (4)$$

The relation between this ψ and ϕ of (3) is $\psi(x) = \phi(x)$ with $J = \{1, \dots, n\}$. One can verify that $\Phi_{C\mathcal{A}_\psi}^* \lambda = 0$.

Example 2.2. Let (C, λ) be a $(2n + 1)$ -dimensional contact manifold, (x, p, z) the canonical coordinates such that $\lambda = dz - p_a dx^a$ with $x = \{x^1, \dots, x^n\}$ and $p = \{p_1, \dots, p_n\}$, and φ a function of p only. Then, the Legendre submanifold \mathcal{A}_φ generated by $-\varphi$ with $\Phi_{C\mathcal{A}_\varphi} : \mathcal{A}_\varphi \rightarrow C$ being the embedding is such that

$$\Phi_{C\mathcal{A}_\varphi} \mathcal{A}_\varphi = \left\{ (x, p, z) \in C \mid x^i = \frac{\partial \varphi}{\partial p_i}, \text{ and } z = p_i \frac{\partial \varphi}{\partial p_i} - \varphi(p), \quad i \in \{1, \dots, n\} \right\}. \quad (5)$$

The relation between this φ and ϕ of (3) is $\varphi(p) = -\phi(p)$ with $I = \{1, \dots, n\}$. One can verify that $\Phi_{C\mathcal{A}_\varphi}^* \lambda = 0$.

In contact geometry the following transform is often used. Note that several conventions exist in the literature.

Definition 2.9. (*Total Legendre(-Fenchel) transform*): Let \mathcal{M} be an n -dimensional manifold, $x = \{x^1, \dots, x^n\}$ coordinates, and ψ a function of x . Then the total Legendre transform of ψ with respect to x is defined to be

$$\mathfrak{L}[\psi](p) := \sup_x [x^a p_a - \psi(x)], \quad (6)$$

where $p = \{p_1, \dots, p_n\}$.

Remark 2.3. If ψ in Example 2.1 is strictly convex, and φ in Example 2.2 is chosen as $\varphi(p) = \mathfrak{L}[\psi](p)$, then it follows that \mathcal{A}_ψ is diffeomorphic to \mathcal{A}_φ (see Ref. [7]).

Introducing some symbols, one can have other equivalent expressions for the Legendre submanifolds (4) and (5). The following definitions were introduced in Ref. [19], and it was shown that the introduced functions are tools to describe contact Hamiltonian vector fields concisely. They are summarized as follows.

Definition 2.10. (Adapted functions): Let (\mathcal{C}, λ) be a $(2n+1)$ -dimensional contact manifold, (x, p, z) canonical coordinates such that $\lambda = dz - p_a dx^a$ with $x = \{x^1, \dots, x^n\}$ and $p = \{p_1, \dots, p_n\}$. In addition let ψ be a function on \mathcal{C} depending on x only, and φ a function on \mathcal{C} depending on p only. Then the functions $\Delta_0^\psi, \{\Delta_1^\psi, \dots, \Delta_n^\psi\} : \mathcal{C} \rightarrow \mathbb{R}$ and $\Delta_\varphi^0, \{\Delta_\varphi^1, \dots, \Delta_\varphi^n\} : \mathcal{C} \rightarrow \mathbb{R}$ such that

$$\Delta_0^\psi(x, z) := \psi(x) - z, \quad \Delta_a^\psi(x, p) := \frac{\partial \psi}{\partial x^a} - p_a, \quad a \in \{1, \dots, n\},$$

$$\Delta_\varphi^0(x, p, z) := x^j p_j - \varphi(p) - z, \quad \Delta_\varphi^a(x, p) := x^a - \frac{\partial \varphi}{\partial p_a}, \quad a \in \{1, \dots, n\},$$

are referred to as adapted functions.

In adapted functions, the local expressions of Legendre submanifolds generated by ψ and those by $-\varphi$ can be written as follows[19].

Proposition 2.2. (Local expressions of Legendre submanifold with adapted functions, [19]): The Legendre submanifold \mathcal{A}_ψ generated by ψ as in (4) is expressed as

$$\Phi_{\mathcal{CA}\psi} \mathcal{A}_\psi = \left\{ (x, p, z) \in \mathcal{C} \mid \Delta_0^\psi = 0 \text{ and } \Delta_1^\psi = \dots = \Delta_n^\psi = 0 \right\},$$

where $\Phi_{\mathcal{CA}\psi} \mathcal{A}_\psi : \mathcal{A}_\psi \rightarrow \mathcal{C}$ is the embedding. Similarly, the Legendre submanifold \mathcal{A}_φ generated by $-\varphi$ as in (5) is expressed as

$$\Phi_{\mathcal{CA}\varphi} \mathcal{A}_\varphi = \left\{ (x, p, z) \in \mathcal{C} \mid \Delta_\varphi^0 = 0 \text{ and } \Delta_\varphi^1 = \dots = \Delta_\varphi^n = 0 \right\},$$

where $\Phi_{\mathcal{CA}\varphi} \mathcal{A}_\varphi : \mathcal{A}_\varphi \rightarrow \mathcal{C}$ is the embedding.

From this proposition, a Legendre submanifold $\Phi_{\mathcal{CA}\psi} \mathcal{A}_\psi$ is a submanifold where the constraints $\Delta_0^\psi = \dots = \Delta_n^\psi$ hold. Thus a vector field on $\Phi_{\mathcal{CA}\psi} \mathcal{A}_\psi$ is the one where relations $\Delta_0^\psi = \dots = \Delta_n^\psi$ hold. This kind of a vector field can be constructed with a restricted contact Hamiltonian vector field. It has been shown in Ref.[19] that contact Hamiltonian vector fields are also written in terms of adapted functions.

Proposition 2.3. (Restricted contact Hamiltonian vector field as the push-forward of a vector field on the Legendre submanifold generated by ψ , [19]): Let $\{F_\psi^1, \dots, F_\psi^n\}$ be a set of functions of x on \mathcal{A}_ψ such that they do not identically vanish, and $\check{X}_\psi^0 \in T_x \mathcal{A}_\psi, (x \in \mathcal{A}_\psi)$ the vector field given as

$$\check{X}_\psi^0 = \dot{x}^a \frac{\partial}{\partial x^a}, \quad \text{where } \dot{x}^a = F_\psi^a(x), \quad (a \in \{1, \dots, n\}).$$

In addition, let $X_\psi^0 := (\Phi_{\mathcal{CA}\psi})_* \check{X}_\psi^0 \in T_\xi \mathcal{A}_\psi^C, (\xi \in \mathcal{A}_\psi^C)$ be the push-forward of \check{X}_ψ^0 , where $\mathcal{A}_\psi^C := \Phi_{\mathcal{CA}\psi} \mathcal{A}_\psi$ with $\Phi_{\mathcal{CA}\psi} : \mathcal{A}_\psi \rightarrow \mathcal{C}$ being the embedding :

$$\begin{aligned} \Phi_{\mathcal{CA}\psi} &: \mathcal{A}_\psi \rightarrow \mathcal{A}_\psi^C, & x &\mapsto (x, p(x), z(x)) \\ (\Phi_{\mathcal{CA}\psi})_* &: T_x \mathcal{A}_\psi \rightarrow T_\xi \mathcal{A}_\psi^C, & \check{X}_\psi^0 &\mapsto X_\psi^0. \end{aligned}$$

Then it follows that

$$X_\psi^0 = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{p}_a \frac{\partial}{\partial p_a} + \dot{z} \frac{\partial}{\partial z}, \quad \text{where } \dot{x}^a = F_\psi^a(x), \quad \dot{p}_a = \frac{d}{dt} \left(\frac{\partial \psi}{\partial x^a} \right), \quad \dot{z} = \frac{d\psi}{dt}. \quad (7)$$

In addition, one has that $X_\psi^0 = X_{h_\psi}|_{h_\psi=0}$. Here X_{h_ψ} is the contact Hamiltonian vector field associated with

$$h_\psi(x, p, z) = \Delta_a(x, p) F_\psi^a(x) + \Gamma_\psi(\Delta_0(x, z)), \quad (8)$$

where Γ_ψ is a function of Δ_0 such that

$$\Gamma_\psi(\Delta_0) = \begin{cases} 0 & \text{for } \Delta_0 = 0 \\ \text{non-zero} & \text{for } \Delta_0 \neq 0 \end{cases}.$$

There exists a counterpart of Proposition 2.3 as follows.

Proposition 2.4. (Restricted contact Hamiltonian vector field as the push-forward of vector fields on the Legendre submanifold generated by $-\varphi$, [19]): Let $\{F_1^\varphi, \dots, F_n^\varphi\}$ be a set of functions of p on \mathcal{A}_φ such that they do not identically vanish, and $\check{X}_\varphi^0 \in T_p \mathcal{A}_\varphi, (p \in \mathcal{A}_\varphi)$ given as

$$\check{X}_\varphi^0 = \dot{p}_a \frac{\partial}{\partial p_a}, \quad \text{where} \quad \dot{p}_a = F_a^\varphi(p).$$

In addition, let $X_\varphi^0 := (\Phi_{\mathcal{CA}_\varphi})_* \check{X}_\varphi^0 \in T_\xi \mathcal{A}_\varphi^C, (\xi \in \mathcal{A}_\varphi^C)$ be the push-forward of \check{X}_φ^0 , where $\mathcal{A}_\varphi^C := \Phi_{\mathcal{CA}_\varphi} \mathcal{A}_\varphi$ with $\Phi_{\mathcal{CA}_\varphi} : \mathcal{A}_\varphi \rightarrow \mathcal{C}$ being the embedding :

$$\begin{aligned} \Phi_{\mathcal{CA}_\varphi} &: \mathcal{A}_\varphi \rightarrow \mathcal{A}_\varphi^C, & x &\mapsto (x(p), p, z(p)) \\ (\Phi_{\mathcal{CA}_\varphi})_* &: T_p \mathcal{A}_\varphi \rightarrow T_\xi \mathcal{A}_\varphi^C, & \check{X}_\varphi^0 &\mapsto X_\varphi^0. \end{aligned}$$

Then it follows that

$$X_\varphi^0 = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{p}_a \frac{\partial}{\partial p_a} + \dot{z} \frac{\partial}{\partial z}, \quad \text{where} \quad \dot{x}_a = \frac{d}{dt} \left(\frac{\partial \varphi}{\partial p_a} \right), \quad \dot{p}^a = F_a^\varphi(p), \quad \dot{z} = p_j F_k^\varphi \frac{\partial^2 \varphi}{\partial p_k \partial p_j}. \quad (9)$$

In addition, one has that $X_\varphi^0 = X_{h_\varphi}|_{h_\varphi=0}$. Here X_{h_φ} is the contact Hamiltonian vector field associated with

$$h_\varphi(x, p) = \Delta^a(x, p) F_a^\varphi(p) + \Gamma^\varphi(\Delta^0), \quad (10)$$

where Γ^φ is a function of Δ^0 such that

$$\Gamma^\varphi(\Delta^0) = \begin{cases} 0 & \text{for } \Delta^0 = 0 \\ \text{non-zero} & \text{for } \Delta^0 \neq 0 \end{cases}.$$

2.2 Fiber bundles

A standard covariant form of Maxwell's equations is formulated on a 4-dimensional pseudo Riemannian manifold, where electromagnetic fields are expressed in terms of a form language. Then the $(3+1)$ -decomposition with respect to an observer of the covariant form of Maxwell's equations gives equations on a 3-dimensional Riemannian manifold. By contrast, a bundle is used in our extended contact geometric description. In this subsection, various definitions and various operators for bundles are introduced to formulate such decomposed Maxwell's equations as a dynamical system in terms of a bundle formalism. The following definition of bundle and the definitions of related objects are used in this paper. More mathematically rigorous definitions can be found in Ref.[20] and so on.

Definition 2.11. (Bundle or fiber bundle): Let \mathcal{B} be a $d_{\mathcal{B}}$ -dimensional manifold with local coordinates $\zeta = \{\zeta_1, \dots, \zeta_{d_{\mathcal{B}}}\}$, \mathcal{F} a $d_{\mathcal{F}}$ -dimensional manifold, \mathcal{M} a $(d_{\mathcal{B}} + d_{\mathcal{F}})$ -dimensional manifold, $\pi : \mathcal{M} \rightarrow \mathcal{B}$ a projection, G a group acting on \mathcal{F} , and $\{U_i\}$ an open covering of \mathcal{B} with $\phi_i : U_i \times \mathcal{F} \rightarrow \pi^{-1}(U_i)$ such that $\pi \phi_i(\zeta, u) = \zeta$. Then the set $(\mathcal{M}, \pi, \mathcal{B})$ or $(\mathcal{M}, \pi, \mathcal{B}, \mathcal{F}, G)$ is referred to as a bundle or a fiber bundle, \mathcal{B} a base space, \mathcal{F} a fiber space, G a structure group, ϕ_i a local trivialization, and \mathcal{M} a total space. Furthermore, let $t_{ij}(\zeta) := \phi_{i,\zeta}^{-1} \circ \phi_{j,\zeta}$ be an element of G ($t_{ij} : U_i \cap U_j \rightarrow G$), where $\phi_{i,\zeta}(u) = \phi_i(\zeta, u)$ for $U_i \cap U_j \neq \emptyset$. Then $\{t_{ij}\}$ are referred to as transition functions.

Definition 2.12. (Trivial bundle and non-trivial bundle): If a transition function for a bundle can be chosen to be identical, then the bundle is referred to as a trivial bundle. Otherwise, the bundle is referred to as a non-trivial bundle.

Non-identical transition functions are used for describing non-trivial bundles. For example, the Möbius band can be constructed with this formulation[20]. In this paper trivial bundles are only considered.

A special class of sub-space of a bundle is considered in this paper, and the definition is given as follows.

Definition 2.13. (Sub-bundle): Let $(\mathcal{M}, \pi, \mathcal{B})$ and $(\mathcal{M}', \pi', \mathcal{B})$ be bundles. If the two conditions,

1. \mathcal{M}' is a submanifold of \mathcal{M} ,
2. $\pi' = \pi|_{\mathcal{M}'}$

are satisfied, then $(\mathcal{M}', \pi', \mathcal{B})$ is referred to as a sub-bundle of $(\mathcal{M}, \pi, \mathcal{B})$.

Definition 2.14. (Section): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle, and $f : \mathcal{B} \rightarrow \mathcal{M}$ a map such that $\pi \circ f = \text{Id}_{\mathcal{B}}$. Then f is referred to as a section. The space of sections is denoted $\Gamma\mathcal{M}$. If $\mathcal{M} = \Omega^0\mathcal{B}$, then $f \in \Gamma\Lambda^0\mathcal{B}$. Similarly the space of q -forms on \mathcal{B} is denoted $\Gamma\Lambda^q\mathcal{B}$, and the space of vector fields on \mathcal{B} as $\Gamma T\mathcal{B}$.

Given a bundle $(\mathcal{M}, \pi, \mathcal{B})$, the space of q -forms on \mathcal{B} and the space of q' -forms on \mathcal{M} can be introduced as follows.

Definition 2.15. (Horizontal forms): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle with $\dim \mathcal{B} = d_{\mathcal{B}}$ and $\dim \mathcal{M} = d_{\mathcal{B}} + d_{\mathcal{F}}$, ζ coordinates for \mathcal{B} with $\zeta = \{\zeta^1, \dots, \zeta^{d_{\mathcal{B}}}\}$, (ζ, u) coordinates for \mathcal{M} with $u = \{u^1, \dots, u^{d_{\mathcal{F}}}\}$, and $\{\alpha_{i_1 \dots i_q}\} \in \Gamma\Lambda^0\mathcal{M}$ some functions. A q -form on the bundle $(\mathcal{M}, \pi, \mathcal{B})$ of the form

$$\alpha_{\mathbb{H}} = \alpha_{i_1 \dots i_q}(\zeta, u) d\zeta^{i_1} \wedge \dots \wedge d\zeta^{i_q},$$

is referred to as a horizontal q -form. The space of horizontal q -forms is denoted $\Gamma\Lambda_{\mathbb{H}}^q\mathcal{M}$.

Definition 2.16. (Vertical forms): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle with $\dim \mathcal{B} = d_{\mathcal{B}}$ and $\dim \mathcal{M} = d_{\mathcal{B}} + d_{\mathcal{F}}$, ζ a set of coordinates for \mathcal{B} , (ζ, u) a set of coordinates for \mathcal{M} with $\zeta = \{\zeta^1, \dots, \zeta^{d_{\mathcal{B}}}\}$ and $u = \{u^1, \dots, u^{d_{\mathcal{F}}}\}$, and $\{\alpha_{i_1 \dots i_q}\} \in \Gamma\Lambda^0\mathcal{M}$ some functions. A q -form on the bundle $(\mathcal{M}, \pi, \mathcal{B})$ of the form

$$\alpha_{\mathbb{V}} = \alpha_{i_1 \dots i_q}(\zeta, u) du^{i_1} \wedge \dots \wedge du^{i_q},$$

is referred to as a vertical q -form. The space of vertical q -forms is denoted $\Gamma\Lambda_{\mathbb{V}}^q\mathcal{M}$. In addition, vertical 0-forms are referred to as vertical functions.

The wedge product of a horizontal q -form and a vertical q' -form can be defined. Then one defines the following.

Definition 2.17. (Mixed form): If a $(q + q')$ -form $\alpha_{\mathbb{M}} \in \Gamma\Lambda^{q+q'}\mathcal{M}$ can be written as

$$\alpha_{\mathbb{M}} = \beta_{\mathbb{H}} \wedge \gamma_{\mathbb{V}},$$

with some $\beta_{\mathbb{H}} \in \Gamma\Lambda_{\mathbb{H}}^q\mathcal{M}$ and $\gamma_{\mathbb{V}} \in \Gamma\Lambda_{\mathbb{V}}^{q'}\mathcal{M}$, then $\alpha_{\mathbb{M}}$ is referred to as a mixed (q, q') -form. The space of mixed (q, q') -forms on \mathcal{M} is denoted $\Gamma\Lambda_{\mathbb{H}, \mathbb{V}}^{q, q'}\mathcal{M}$.

Definition 2.18. (Vertical derivative): Let $\alpha_{\mathbb{V}} \in \Gamma\Lambda_{\mathbb{V}}^q\mathcal{M}$ be a vertical q -form whose local expression is

$$\alpha_{\mathbb{V}} = \alpha_{i_1 \dots i_q}(\zeta, u) du^{i_1} \wedge \dots \wedge du^{i_q}.$$

The operator $d_{\mathbb{V}} : \Gamma\Lambda_{\mathbb{V}}^q\mathcal{M} \rightarrow \Gamma\Lambda_{\mathbb{V}}^{q+1}\mathcal{M}$ whose action is such that

$$d_{\mathbb{V}}\alpha_{\mathbb{V}} = \frac{\partial \alpha_{i_1 \dots i_q}(\zeta, u)}{\partial u^{i_0}} du^{i_0} \wedge du^{i_1} \wedge \dots \wedge du^{i_q},$$

is referred to as the vertical derivative or the vertical exterior derivative.

Definition 2.19. (Functional): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle, $\alpha_{\mathbb{H}} \in \Gamma\Lambda_{\mathbb{H}}^q\mathcal{M}$ a horizontal q -form, $h \in \Gamma\Lambda_{\mathbb{V}}^0\mathcal{M}$ a vertical 0-form, and $\mathcal{B}_0 \subseteq \mathcal{B}$ a q -dimensional space. The integral over \mathcal{B}_0

$$\tilde{h}_{\mathcal{B}_0} = \int_{\mathcal{B}_0} h \alpha_{\mathbb{H}},$$

is referred to as a functional. The space of functionals is denoted as $\Gamma F\mathcal{M}$.

The functional derivative has been used in the infinite dimensional Hamiltonian formulation of Maxwell's equations[23]. In this paper this derivative can also be used to describe Maxwell's equations as a dynamical system.

Definition 2.20. (*Functional derivative*): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle, $\alpha_{\mathbb{M}}$ a mixed $(q, 0)$ -form on a q' -dimensional submanifold of \mathcal{B} , $\mathcal{B}_0 \subseteq \mathcal{B}$ a q' -dimensional subspace, $\tilde{h}_{\mathcal{B}_0}$ a functional depending on $\alpha_{\mathbb{M}}$, and $\eta \in \mathbb{R}$ a constant. Then the mixed $(q' - q, 0)$ -form $\delta \tilde{h}_{\mathcal{B}_0} / \delta \alpha_{\mathbb{M}} \in \Gamma \Lambda_{\mathbb{H}, \mathbb{V}}^{q' - q, 0} \mathcal{M}$ that is uniquely obtained by

$$\tilde{h}_{\mathcal{B}_0} [\alpha_{\mathbb{M}} + \eta \alpha'_{\mathbb{M}}] = \tilde{h}_{\mathcal{B}_0} [\alpha_{\mathbb{M}}] + \eta \int_{\mathcal{B}_0} \frac{\delta \tilde{h}_{\mathcal{B}_0}}{\delta \alpha_{\mathbb{M}}} \wedge \alpha'_{\mathbb{M}} + \mathcal{O}(\eta^2), \quad \forall \alpha'_{\mathbb{M}} \in \Gamma \Lambda_{\mathbb{H}, \mathbb{V}}^{q, 0} \mathcal{M}$$

is referred to as the functional derivative of $\tilde{h}_{\mathcal{B}_0}$ with respect to $\alpha_{\mathbb{M}}$.

Similar to the case of forms, the space of vector fields on \mathcal{B} and the space of vector fields on \mathcal{M} can be introduced as follows.

Definition 2.21. (*Horizontal vector field*): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle with $\dim \mathcal{B} = d_{\mathcal{B}}$ and $\dim \mathcal{M} = d_{\mathcal{B}} + d_{\mathcal{F}}$, ζ a set of coordinates for \mathcal{B} with $\zeta = \{\zeta^1, \dots, \zeta^{d_{\mathcal{B}}}\}$, (ζ, u) a set of coordinates for \mathcal{M} with $u = \{u^1, \dots, u^{d_{\mathcal{F}}}\}$, and $\{Y_1, \dots, Y_{d_{\mathcal{B}}}\} \in \Gamma \Lambda^0 \mathcal{M}$ some functions. A vector field on the bundle $(\mathcal{M}, \pi, \mathcal{B})$ of the form

$$Y_{\mathbb{H}} = Y_i(\zeta, u) \frac{\partial}{\partial \zeta^i},$$

is referred to as a horizontal vector field. The space of horizontal vector fields is denoted as $\Gamma T_{\mathbb{H}} \mathcal{M}$.

Remark 2.4. The dual of a horizontal vector field is a horizontal 1-form.

Definition 2.22. (*Vertical vector field*): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle with $\dim \mathcal{B} = d_{\mathcal{B}}$ and $\dim \mathcal{M} = d_{\mathcal{B}} + d_{\mathcal{F}}$, ζ a set of coordinates for \mathcal{B} with $\zeta = \{\zeta^1, \dots, \zeta^{d_{\mathcal{B}}}\}$, (ζ, u) a set of coordinates for \mathcal{M} with $u = \{u^1, \dots, u^{d_{\mathcal{F}}}\}$, and $\{Y_1, \dots, Y_{d_{\mathcal{F}}}\} \in \Gamma \Lambda^0 \mathcal{M}$ some functions. A vector field on the bundle $(\mathcal{M}, \pi, \mathcal{B})$ of the form

$$Y_{\mathbb{V}} = Y_i(\zeta, u) \frac{\partial}{\partial u^i},$$

is referred to as a vertical vector field. The space of vertical vector fields is denoted as $\Gamma T_{\mathbb{V}} \mathcal{M}$.

Remark 2.5. The dual of a vertical vector field is a vertical 1-form.

The action of the interior product with respect to a vertical vector field $Y_{\mathbb{V}}$ for a vertical q -form $\alpha_{\mathbb{V}}$ is denoted $\iota_{Y_{\mathbb{V}}} \alpha_{\mathbb{V}}$ and is similar to the action of a vector field Y to a q -form α , $\iota_Y \alpha$.

Definition 2.23. (*Interior product associated with a vertical vector field for vertical form*): Let $(\mathcal{M}, \pi, \mathcal{B})$ be a bundle with $\dim \mathcal{B} = d_{\mathcal{B}}$ and $\dim \mathcal{M} = d_{\mathcal{B}} + d_{\mathcal{F}}$, ζ a set of coordinates for \mathcal{B} with $\zeta = \{\zeta^1, \dots, \zeta^{d_{\mathcal{B}}}\}$, $\beta_{\mathbb{H}} \in \Gamma \Lambda_{\mathbb{H}}^q \mathcal{M}$ a horizontal q -form, $\gamma_{\mathbb{V}} \in \Gamma \Lambda_{\mathbb{V}}^{q'} \mathcal{M}$ a vertical q' -form, $Y_{\mathbb{V}} \in \Gamma T_{\mathbb{V}} \mathcal{M}$ a vertical vector field, and $\alpha_{\mathbb{M}} \in \Gamma \Lambda_{\mathbb{H}, \mathbb{V}}^{q, q'} \mathcal{M}$ a mixed (q, q') -form written as

$$\alpha_{\mathbb{M}} = \gamma_{\mathbb{V}} \wedge \beta_{\mathbb{H}}.$$

Then the action of $\iota_{Y_{\mathbb{V}}}$ to $\alpha_{\mathbb{M}}$, $\iota_{Y_{\mathbb{V}}} : \Gamma \Lambda_{\mathbb{H}, \mathbb{V}}^{q, q'} \mathcal{M} \rightarrow \Gamma \Lambda_{\mathbb{H}, \mathbb{V}}^{q, q' - 1} \mathcal{M}$ is defined as

$$\iota_{Y_{\mathbb{V}}} \alpha_{\mathbb{M}} = (\iota_{Y_{\mathbb{V}}} \gamma_{\mathbb{V}}) \wedge \beta_{\mathbb{H}}.$$

3 Contact manifold over base space

In this paper a contact manifold over a base space is treated as a bundle.

Definition 3.1. (*Contact manifold over a base space*): Let \mathcal{B} be a $d_{\mathcal{B}}$ -dimensional manifold, $(\mathcal{K}, \pi, \mathcal{B})$ a bundle over the base space \mathcal{B} , the fiber space $\pi^{-1}(\zeta)$ at a point ζ of \mathcal{B} a $(2n+1)$ -dimensional manifold \mathcal{C}_{ζ} , $\mathcal{K} = \bigcup_{\zeta \in \mathcal{B}} \mathcal{C}_{\zeta}$, and the structure group G a contact transformation group. If \mathcal{K} carries a vertical form $\lambda_{\mathbb{V}}$ such that

$$\lambda_{\mathbb{V}} \wedge \underbrace{d_{\mathbb{V}}\lambda_{\mathbb{V}} \wedge \cdots \wedge d_{\mathbb{V}}\lambda_{\mathbb{V}}}_n \neq 0, \quad \text{at each point of } \pi^{-1}(\zeta) \text{ at each point } \zeta \text{ of } \mathcal{B}$$

then \mathcal{C}_{ζ} is referred to as a $((2n+1)$ -dimensional) contact manifold on the fiber space $\pi^{-1}(\zeta)$, $(\zeta \in \mathcal{B})$, the quadruplet $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ is referred to as a $((2n+1)$ -dimensional) contact manifold over the base space \mathcal{B} , and $\lambda_{\mathbb{V}}$ a contact vertical form.

In this paper we only consider trivial bundles, then the transition function is identical since this simple case is enough for our contact geometric formulation of Maxwell's equations without source in media. The contact geometry of the vertical space is the same as the standard contact geometry. Thus, all of the definitions and theorems for the standard contact geometry can be brought to vertical spaces. They are shown below.

At each base point ζ of \mathcal{B} , one has Darboux's theorem for $\pi^{-1}(\zeta)$. Therefore one has the following.

Theorem 3.1. (*Existence of Darboux coordinates on fiber space*): For a $(2n+1)$ -dimensional contact manifold over a base space $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$, there exist local coordinates (x, p, z) for $\pi^{-1}(\zeta)$ with $x = \{x^1, \dots, x^n\}$ and $p = \{p_1, \dots, p_n\}$ in which $\lambda_{\mathbb{M}}^q \in \Gamma\Lambda_{\mathbb{H}, \mathbb{V}}^{q,1}\mathcal{K}$ has the form

$$\lambda_{\mathbb{M}}^q = \rho^q \wedge \lambda_{\mathbb{V}}, \quad \text{where} \quad \lambda_{\mathbb{V}} = d_{\mathbb{V}}z - p_a d_{\mathbb{V}}x^a,$$

with some $\rho^q \in \Gamma\Lambda_{\mathbb{H}}^q\mathcal{K}$ being nowhere vanishing.

Definition 3.2. (*Canonical coordinates or Darboux coordinates*): The $(2n+1)$ coordinates introduced in Theorem 3.1 are referred to as the canonical coordinates for a fiber space or the Darboux coordinates for a fiber space.

Definition 3.3. (*Canonical contact mixed form and canonical contact vertical form*): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} , and ρ^q a nowhere vanishing horizontal q -form. A mixed $(q, 1)$ -form $\lambda_{\mathbb{M}}^q \in \Gamma\Lambda_{\mathbb{H}, \mathbb{V}}^{q,1}\mathcal{K}$ written as

$$\lambda_{\mathbb{M}}^q = \rho^q \wedge \lambda_{\mathbb{V}}, \quad \text{where} \quad \lambda_{\mathbb{V}} = d_{\mathbb{V}}z - p_a d_{\mathbb{V}}x^a,$$

is referred to as the canonical contact mixed $(q, 1)$ -form associated with ρ^q , and $\lambda_{\mathbb{V}} \in \Gamma\Lambda_{\mathbb{V}}^1\mathcal{K}$ the canonical contact vertical form.

Remark 3.1. With the canonical contact vertical form $\lambda_{\mathbb{V}} \in \Gamma\Lambda_{\mathbb{V}}^1\mathcal{K}$ and a nowhere vanishing horizontal $d_{\mathcal{B}}$ -form $\rho^{d_{\mathcal{B}}}$, the mixed $(d_{\mathcal{B}}, 2n+1)$ -form

$$\rho^{d_{\mathcal{B}}} \wedge \lambda_{\mathbb{V}} \wedge \underbrace{d_{\mathbb{V}}\lambda_{\mathbb{V}} \wedge \cdots \wedge d_{\mathbb{V}}\lambda_{\mathbb{V}}}_n \in \Gamma\Lambda_{\mathbb{H}, \mathbb{V}}^{d_{\mathcal{B}}, 2n+1}\mathcal{K}$$

is a volume-form on \mathcal{K} .

Definition 3.4. (*Reeb vertical vector field*): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} , and $\mathcal{R}_{\mathbb{V}}$ a vertical vector field on \mathcal{K} . If $\mathcal{R}_{\mathbb{V}}$ satisfies

$$\iota_{\mathcal{R}_{\mathbb{V}}}\lambda_{\mathbb{V}} = 1, \quad \text{and} \quad \iota_{\mathcal{R}_{\mathbb{V}}}d_{\mathbb{V}}\lambda_{\mathbb{V}} = 0,$$

then $\mathcal{R}_{\mathbb{V}}$ is referred to as the Reeb vertical vector field on \mathcal{K} .

Proposition 3.1. (Coordinate expression of the Reeb vertical vector field): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} , and $\mathcal{R}_{\mathbb{V}}$ the Reeb vertical vector field on \mathcal{K} , and (x, p, z) the canonical coordinates such that $\lambda_{\mathbb{V}} = d_{\mathbb{V}}z - p_a d_{\mathbb{V}}x^a$. Then the coordinate expression of the Reeb vector field is

$$\mathcal{R}_{\mathbb{V}} = \frac{\partial}{\partial z}.$$

Definition 3.5. (Contact Hamiltonian vertical vector field): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} with $\dim \mathcal{B} = d_{\mathcal{B}}$, $\mathcal{B}_0 \subseteq \mathcal{B}$ a $d_{\mathcal{B}}$ -dimensional space, $\rho^{d_{\mathcal{B}}} \in \Gamma \Lambda_{\mathbb{H}}^{d_{\mathcal{B}}} \mathcal{K}$ a nowhere vanishing horizontal form, $\mathcal{R}_{\mathbb{V}}$ the Reeb vertical vector field on \mathcal{K} , $\tilde{h} \in \Gamma FK$ the functional given by

$$\tilde{h} = \int_{\mathcal{B}_0} h \rho^{d_{\mathcal{B}}},$$

with some $h \in \Gamma \Lambda_{\mathbb{V}}^0 \mathcal{K}$, and $X_{\tilde{h}}$ a vertical vector field on \mathcal{K} . If $X_{\tilde{h}}$ satisfies

$$\iota_{X_{\tilde{h}}} \lambda_{\mathbb{V}} = h \quad \text{and} \quad \iota_{X_{\tilde{h}}} d_{\mathbb{V}} \lambda_{\mathbb{V}} = - (d_{\mathbb{V}} h - (\mathcal{R}_{\mathbb{V}} h) \lambda_{\mathbb{V}}), \quad (11)$$

then $X_{\tilde{h}}$ is referred to as the contact Hamiltonian vertical vector field, \tilde{h} a contact Hamiltonian functional, and h a contact Hamiltonian vertical function.

Remark 3.2. With the Cartan formula, one has that $\mathcal{L}_{X_{\tilde{h}}} \lambda_{\mathbb{V}} = (\mathcal{R}_{\mathbb{V}} h) \lambda_{\mathbb{V}}$. Thus, $\mathcal{L}_{X_{\tilde{h}}} \lambda_{\mathbb{M}}^{d_{\mathcal{B}}} = \rho^{d_{\mathcal{B}}} \wedge \mathcal{L}_{X_{\tilde{h}}} \lambda_{\mathbb{V}} = (\mathcal{R}_{\mathbb{V}} h) \lambda_{\mathbb{M}}^{d_{\mathcal{B}}}$.

In the following the coordinate expression of a contact Hamiltonian vertical vector field is shown.

Proposition 3.2. (Coordinate expression of a contact vertical Hamiltonian vector field): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} with $\dim \mathcal{B} = d_{\mathcal{B}}$, $\mathcal{B}_0 \subseteq \mathcal{B}$ a $d_{\mathcal{B}}$ -dimensional space, $\rho^{d_{\mathcal{B}}} \in \Gamma \Lambda_{\mathbb{H}}^{d_{\mathcal{B}}} \mathcal{K}$ a nowhere vanishing form, (x, p, z) the canonical coordinates for the fiber space such that $\lambda_{\mathbb{V}} = d_{\mathbb{V}}z - p_a d_{\mathbb{V}}x^a$ with $x = \{x^1, \dots, x^n\}$ and $p = \{p_1, \dots, p_n\}$, \tilde{h} the contact Hamiltonian functional given by

$$\tilde{h} = \int_{\mathcal{B}_0} h \rho^{d_{\mathcal{B}}},$$

with some $h \in \Gamma \Lambda_{\mathbb{V}}^0 \mathcal{K}$ depending on (x, p, z) , and $X_{\tilde{h}}$ the contact Hamiltonian vertical vector field on \mathcal{K} .

Then, the canonical coordinate expression of (11) is given as

$$X_{\tilde{h}} = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{p}_a \frac{\partial}{\partial p_a} + \dot{z} \frac{\partial}{\partial z},$$

where

$$\dot{x}^a = - \frac{\partial h}{\partial p_a}, \quad \dot{p}_a = \frac{\partial h}{\partial x^a} + p_a \frac{\partial h}{\partial z}, \quad \dot{z} = h - p_a \frac{\partial h}{\partial p_a}, \quad (12)$$

or equivalently,

$$\dot{x}^a = - \frac{\delta \tilde{h}}{\delta p_a}, \quad \dot{p}_a = \frac{\delta \tilde{h}}{\delta x^a} + p_a \frac{\delta \tilde{h}}{\delta z}, \quad \dot{z} = h - p_a \frac{\delta \tilde{h}}{\delta p_a}.$$

Remark 3.3. The coordinate expression (12) is formally the same as that of (2).

Analogous to Definition 2.7, Legendre submanifold on a bundle is defined as follows.

Definition 3.6. (Legendre submanifold of vertical space and that of fiber space) : Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} . If \mathcal{A}_{ζ} is a maximal dimensional integral submanifold of $\lambda_{\mathbb{V}}$ on $\pi^{-1}(\zeta)$, ($\zeta \in \mathcal{B}$), then \mathcal{A}_{ζ} is referred to as a Legendre submanifold in the fiber space $\pi^{-1}(\zeta)$, and $\mathcal{A}^{\mathcal{K}} = \bigcup_{\zeta \in \mathcal{B}} \mathcal{A}_{\zeta}$ a Legendre submanifold of the fiber space.

An analogous theorem from Theorem 2.3 holds for our bundles. Then examples of Legendre submanifolds on fiber spaces are as follows.

Example 3.1. Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a $(2n+1)$ -dimensional contact manifold over a base space \mathcal{B} with $\dim \mathcal{B} = d_{\mathcal{B}}$, (x, p, z) the canonical coordinates for the fiber space such that $\lambda_{\mathbb{V}} = d_{\mathbb{V}}z - p_a d_{\mathbb{V}}x^a$ with $x = \{x^1, \dots, x^n\}$ and $p = \{p_1, \dots, p_n\}$, $\psi \in \Gamma \Lambda_{\mathbb{V}}^0 \mathcal{K}$ a vertical function of x , $\rho^{d_{\mathcal{B}}}$ a nowhere vanishing horizontal $d_{\mathcal{B}}$ -form, $\mathcal{B}_0 \subseteq \mathcal{B}$ a $d_{\mathcal{B}}$ -dimensional space, and $\tilde{\psi}_{\mathcal{B}_0} \in \Gamma F\mathcal{K}$ the functional

$$\tilde{\psi}_{\mathcal{B}_0} = \int_{\mathcal{B}_0} \psi \rho^{d_{\mathcal{B}}}.$$

Then, the Legendre submanifold $\mathcal{A}_{\zeta\psi}$ generated by ψ in $\pi^{-1}(\zeta)$, $(\zeta \in \mathcal{B})$ with $\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\psi}} : \mathcal{A}_{\zeta\psi} \rightarrow \mathcal{C}_{\zeta}$ being the embedding is such that

$$\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\psi}} \mathcal{A}_{\zeta\psi} = \left\{ (x, p, z) \in \mathcal{C}_{\zeta} \mid p_j = \frac{\partial \psi}{\partial x^j}, \text{ and } z = \psi(x), \quad j \in \{1, \dots, n\} \right\}. \quad (13)$$

This can also be written as

$$\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\psi}} \mathcal{A}_{\zeta\psi} = \left\{ (x, p, z) \in \mathcal{C}_{\zeta} \mid p_j = \frac{\delta \tilde{\psi}_{\mathcal{B}_0}}{\delta x^j}, \text{ and } z = \psi(x), \quad j \in \{1, \dots, n\} \right\}.$$

In addition, $(\mathcal{A}_{\psi}^{\mathcal{K}}, \pi|_{\mathcal{A}_{\psi}^{\mathcal{K}}}, \mathcal{B})$ is a sub-bundle of $(\mathcal{K}, \pi, \mathcal{B})$, where

$$\mathcal{A}_{\psi}^{\mathcal{K}} = \bigcup_{\zeta \in \mathcal{B}} \Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\psi}} \mathcal{A}_{\zeta\psi}.$$

Example 3.2. Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a $(2n+1)$ -dimensional contact manifold over a base space \mathcal{B} , (x, p, z) the canonical coordinates for the fiber space such that $\lambda_{\mathbb{V}} = d_{\mathbb{V}}z - p_a d_{\mathbb{V}}x^a$ with $x = \{x^1, \dots, x^n\}$ and $p = \{p_1, \dots, p_n\}$, $\varphi \in \Gamma \Lambda_{\mathbb{V}}^0 \mathcal{K}$ a vertical function of p , $\rho^{d_{\mathcal{B}}}$ a nowhere vanishing horizontal $d_{\mathcal{B}}$ -form, $\mathcal{B}_0 \subseteq \mathcal{B}$ a $d_{\mathcal{B}}$ -dimensional space, and $\tilde{\varphi}_{\mathcal{B}_0} \in \Gamma F\mathcal{K}$ the functional

$$\tilde{\varphi}_{\mathcal{B}_0} = \int_{\mathcal{B}_0} \varphi \rho^{d_{\mathcal{B}}}.$$

Then, the Legendre submanifold $\mathcal{A}_{\zeta\varphi}$ generated by $-\varphi$ in $\pi^{-1}(\zeta)$, $(\zeta \in \mathcal{B})$ with $\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\varphi}} : \mathcal{A}_{\zeta\varphi} \rightarrow \mathcal{C}_{\zeta}$ being the embedding is such that

$$\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\varphi}} \mathcal{A}_{\zeta\varphi} = \left\{ (x, p, z) \in \mathcal{C}_{\zeta} \mid x_i = \frac{\partial \varphi}{\partial p_i}, \text{ and } z = p_i \frac{\partial \varphi}{\partial p_i} - \varphi(p), \quad i \in \{1, \dots, n\} \right\}. \quad (14)$$

This can also be written as

$$\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\varphi}} \mathcal{A}_{\zeta\varphi} = \left\{ (x, p, z) \in \mathcal{C}_{\zeta} \mid x_i = \frac{\delta \tilde{\varphi}_{\mathcal{B}_0}}{\delta p_i}, \text{ and } z = p_i \frac{\delta \tilde{\varphi}_{\mathcal{B}_0}}{\delta p_i} - \varphi(p), \quad i \in \{1, \dots, n\} \right\}.$$

In addition, $(\mathcal{A}_{\varphi}^{\mathcal{K}}, \pi|_{\mathcal{A}_{\varphi}^{\mathcal{K}}}, \mathcal{B})$ is a sub-bundle of $(\mathcal{K}, \pi, \mathcal{B})$, where

$$\mathcal{A}_{\varphi}^{\mathcal{K}} = \bigcup_{\zeta \in \mathcal{B}} \Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\varphi}} \mathcal{A}_{\zeta\varphi}.$$

Although the following could not be commonly used in the literature, the total Legendre transform of a functional is defined as follows in this paper.

Definition 3.7. (Total Legendre transform of functional): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a contact manifold over a base space \mathcal{B} with $\dim \mathcal{B} = d_{\mathcal{B}}$, $\mathcal{B}_0 \subseteq \mathcal{B}$ a $d_{\mathcal{B}}$ -dimensional space, $\rho^{d_{\mathcal{B}}} \in \Gamma \Lambda_{\mathbb{H}}^{d_{\mathcal{B}}} \mathcal{K}$ a nowhere vanishing horizontal form, $\psi \in \Gamma \Lambda_{\mathbb{V}}^0 \mathcal{K}$ a vertical 0-form, $\tilde{\psi}_{\mathcal{Z}_0}$ a functional such that

$$\tilde{\psi}_{\mathcal{Z}_0} = \int_{\mathcal{Z}_0} \psi \rho^{d_{\mathcal{B}}}.$$

Then, the total Legendre transform of $\Psi_{\mathcal{Z}_0}$ is defined as

$$\tilde{\psi}_{\mathcal{Z}_0}^* = \int_{\mathcal{Z}_0} \mathfrak{L}[\psi] \rho^{d_{\mathcal{B}}},$$

where $\mathfrak{L}[\psi]$ is the total Legendre transform of ψ .

As shown in Propositions 2.3 and 2.4, vector fields on Legendre submanifolds of contact manifolds are concisely written as contact Hamiltonian vector fields with adapted functions introduced in Definition 2.10 for the standard contact geometry. Also, for contact geometry on fiber spaces, similar functions can be defined as follows.

Definition 3.8. (Adapted functions on fiber space): Let $(\mathcal{K}, \lambda_{\mathbb{V}}, \pi, \mathcal{B})$ be a $(2n+1)$ -dimensional contact manifold over a base space \mathcal{B} , (x, p, z) canonical coordinates for $\pi^{-1}(\zeta)$, $(\zeta \in \mathcal{B})$ such that $\lambda_{\mathbb{V}} = d_{\mathbb{V}}z - p_a d_{\mathbb{V}}x^a$ with $x = \{x^1, \dots, x^n\}$ and $p = \{p_1, \dots, p_n\}$, and $\mathcal{K} = \bigcup_{\zeta \in \mathcal{B}} \mathcal{C}_{\zeta}$. In addition let ψ be a vertical function on \mathcal{C}_{ζ} depending on x , and φ a vertical function on \mathcal{C}_{ζ} depending on p . Then the functions $\Delta_0^{\zeta\psi}, \{\Delta_1^{\zeta\psi}, \dots, \Delta_n^{\zeta\psi}\} : \mathcal{C}_{\zeta} \rightarrow \mathbb{R}$, and $\Delta_{\zeta\varphi}^0, \{\Delta_{\zeta\varphi}^1, \dots, \Delta_{\zeta\varphi}^n\} : \mathcal{C}_{\zeta} \rightarrow \mathbb{R}$ such that

$$\Delta_0^{\zeta\psi}(x, z) := \psi(x) - z, \quad \Delta_a^{\zeta\psi}(x, p) := \frac{\partial \psi}{\partial x^a} - p_a, \quad a \in \{1, \dots, n\}.$$

$$\Delta_{\zeta\varphi}^0(x, p, z) := x^j p_j - \varphi(p) - z, \quad \Delta_{\zeta\varphi}^a(x, p) := x^a - \frac{\partial \varphi}{\partial p_a}, \quad a \in \{1, \dots, n\}.$$

are referred to as adapted functions on the fiber space.

Similar to Proposition 2.2, one has the following.

Proposition 3.3. (Local expressions of Legendre submanifold with adapted functions): The Legendre submanifold \mathcal{A}_{ψ} generated by ψ in $\pi^{-1}(\zeta)$ as (13) is expressed as

$$\mathcal{A}_{\zeta\psi}^{\mathcal{C}} := \Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\psi}} \mathcal{A}_{\zeta\psi} = \left\{ (x, p, z) \in \mathcal{C}_{\zeta} \mid \Delta_0^{\zeta\psi} = 0 \text{ and } \Delta_1^{\zeta\psi} = \dots = \Delta_n^{\zeta\psi} = 0 \right\}, \quad (15)$$

where $\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\psi}} : \mathcal{A}_{\zeta\psi} \rightarrow \mathcal{C}_{\zeta}$ is the embedding. Similarly, the Legendre submanifold \mathcal{A}_{φ} generated by $-\varphi$ in $\pi^{-1}(\zeta)$ as (14) is expressed as

$$\mathcal{A}_{\zeta\varphi}^{\mathcal{C}} := \Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\varphi}} \mathcal{A}_{\zeta\varphi} = \left\{ (x, p, z) \in \mathcal{C}_{\zeta} \mid \Delta_{\zeta\varphi}^0 = 0 \text{ and } \Delta_{\zeta\varphi}^1 = \dots = \Delta_{\zeta\varphi}^n = 0 \right\}, \quad (16)$$

where $\Phi_{\mathcal{C}_{\zeta}\mathcal{A}_{\zeta\varphi}} : \mathcal{A}_{\zeta\varphi} \rightarrow \mathcal{C}_{\zeta}$ is the embedding.

Contact Hamiltonian vertical vector fields are also written in terms of adapted functions on fiber spaces.

Proposition 3.4. (Restricted contact Hamiltonian vertical vector field as the push-forward of a vector field on the Legendre submanifold generated by ψ): Let $\{F_{\psi}^{\zeta,1}, \dots, F_{\psi}^{\zeta,n}\}$ be a set of functions of x on $\mathcal{A}_{\zeta\psi}$ such that they do not identically vanish, and $\tilde{X}_{\zeta\psi}^0 \in T_x \mathcal{A}_{\zeta\psi}$, $(x \in \mathcal{A}_{\zeta\psi})$ the vector field given as

$$\tilde{X}_{\zeta\psi}^0 = \dot{x}^a \frac{\partial}{\partial x^a}, \quad \text{where } \dot{x}^a = F_{\zeta\psi}^a(x), \quad (a \in \{1, \dots, n\}).$$

In addition, let $X_{\zeta\psi}^0 := (\Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\psi}})_* \tilde{X}_{\zeta\psi}^0 \in T_\xi \mathcal{A}_{\zeta\psi}^{\mathcal{C}}, (\xi \in \mathcal{A}_{\zeta\psi}^{\mathcal{C}})$ be the push-forward of $\tilde{X}_{\zeta\psi}^0$, where $\mathcal{A}_{\zeta\psi}^{\mathcal{C}} := \Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\psi}} \mathcal{A}_{\zeta\psi}$ with $\Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\psi}} : \mathcal{A}_{\zeta\psi} \rightarrow \mathcal{C}_\zeta$ being the embedding :

$$\begin{aligned} \Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\psi}} &: \mathcal{A}_{\zeta\psi} \rightarrow \mathcal{A}_{\zeta\psi}^{\mathcal{C}}, & x &\mapsto (x, p(x), z(x)), \quad \text{on } \pi^{-1}(\zeta), \\ (\Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\psi}})_* &: T_x \mathcal{A}_{\zeta\psi} \rightarrow T_\xi \mathcal{A}_{\zeta\psi}^{\mathcal{C}}, & \tilde{X}_{\zeta\psi}^0 &\mapsto X_{\zeta\psi}^0. \end{aligned}$$

Then it follows that

$$X_{\zeta\psi}^0 = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{p}_a \frac{\partial}{\partial p_a} + \dot{z} \frac{\partial}{\partial z}, \quad \text{where } \dot{x}^a = F_{\zeta\psi}^a(x), \quad \dot{p}_a = \frac{d}{dt} \left(\frac{\partial \psi}{\partial x^a} \right), \quad \dot{z} = \frac{d\psi}{dt}. \quad (17)$$

In addition, one has that $X_{\zeta\psi}^0 = X_{\tilde{h}_\psi}|_{\tilde{h}_\psi=0}$. Here $X_{\tilde{h}_\psi}$ is the contact Hamiltonian vertical vector field associated with

$$h_\psi(x, p, z) = \Delta_a^{\zeta\psi}(x, p) F_{\zeta\psi}^a(x) + \Gamma_{\zeta\psi}(\Delta_0^{\zeta\psi}(x, z)), \quad (18)$$

where $\Gamma_{\zeta\psi}$ is a function of $\Delta_0^{\zeta\psi}$ such that

$$\Gamma_{\zeta\psi}(\Delta_0^{\zeta\psi}) = \begin{cases} 0 & \text{for } \Delta_0^{\zeta\psi} = 0 \\ \text{non-zero} & \text{for } \Delta_0^{\zeta\psi} \neq 0 \end{cases}.$$

There exists a counterpart of Proposition 3.4, that is given as follows.

Proposition 3.5. (Restricted contact Hamiltonian vertical vector field as the push-forward of a vector field on the Legendre submanifold generated by $-\varphi$): Let $\{F_{\zeta,1}^\varphi, \dots, F_{\zeta,n}^\varphi\}$ be a set of functions of p on $\mathcal{A}_{\zeta\varphi}$ such that they do not identically vanish, and $\tilde{X}_0^{\zeta\varphi} \in T_p \mathcal{A}_{\zeta\varphi}, (p \in \mathcal{A}_{\zeta\varphi})$ the vector field given as

$$\tilde{X}_0^{\zeta\varphi} = \dot{p}_a \frac{\partial}{\partial p_a}, \quad \text{where } \dot{p}_a = F_a^{\zeta\varphi}(p), \quad (a \in \{1, \dots, n\}).$$

In addition, let $X_0^{\zeta\varphi} := (\Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\varphi}})_* \tilde{X}_0^{\zeta\varphi} \in T_\xi \mathcal{A}_{\zeta\varphi}^{\mathcal{C}}, (\xi \in \mathcal{A}_{\zeta\varphi}^{\mathcal{C}})$ be the push-forward of $\tilde{X}_0^{\zeta\varphi}$, where $\mathcal{A}_{\zeta\varphi}^{\mathcal{C}} := \Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\varphi}} \mathcal{A}_{\zeta\varphi}$ with $\Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\varphi}} : \mathcal{A}_{\zeta\varphi} \rightarrow \mathcal{C}_\zeta$ being the embedding :

$$\begin{aligned} \Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\varphi}} &: \mathcal{A}_{\zeta\varphi} \rightarrow \mathcal{A}_{\zeta\varphi}^{\mathcal{C}}, & p &\mapsto (x(p), p, z(p)), \quad \text{on } \pi^{-1}(\zeta), \\ (\Phi_{\mathcal{C}_\zeta \mathcal{A}_{\zeta\varphi}})_* &: T_p \mathcal{A}_{\zeta\varphi} \rightarrow T_\xi \mathcal{A}_{\zeta\varphi}^{\mathcal{C}}, & \tilde{X}_0^{\zeta\varphi} &\mapsto X_0^{\zeta\varphi}. \end{aligned}$$

Then, it follows that

$$X_0^{\zeta\varphi} = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{p}_a \frac{\partial}{\partial p_a} + \dot{z} \frac{\partial}{\partial z}, \quad \text{where } \dot{x}^a = \frac{d}{dt} \left(\frac{\partial \varphi}{\partial p_a} \right), \quad \dot{p}_a = F_a^{\zeta\varphi}(p), \quad \dot{z} = p_j F_k^{\zeta\varphi} \frac{\partial^2 \varphi}{\partial p_k \partial p_j}. \quad (19)$$

In addition, one has that $X_0^{\zeta\varphi} = X_{\tilde{h}_\varphi}|_{\tilde{h}_\varphi=0}$. Here $X_{\tilde{h}_\varphi}$ is the contact Hamiltonian vertical vector field associated with

$$h_\varphi(x, p, z) = \Delta_a^{\zeta\varphi}(x, p) F_a^{\zeta\varphi}(p) + \Gamma^{\zeta\varphi}(\Delta_{\zeta\varphi}^0(x, p, z)), \quad (20)$$

where $\Gamma^{\zeta\varphi}$ is a function of $\Delta_{\zeta\varphi}^0$ such that

$$\Gamma^{\zeta\varphi}(\Delta_{\zeta\varphi}^0) = \begin{cases} 0 & \text{for } \Delta_{\zeta\varphi}^0 = 0 \\ \text{non-zero} & \text{for } \Delta_{\zeta\varphi}^0 \neq 0 \end{cases}.$$

4 Maxwell's equations without source

In this section the $(3+1)$ -decomposed Maxwell's equations without source in terms of a form language is summarized. This formulation is standard and can be found in the literature[21].

4.1 Three-dimensional Riemannian manifold

To discuss (3+1)-decomposed Maxwell's equations in our extended framework of contact geometry, a bundle will be used. In this extended framework, the base space is a 3-dimensional Riemannian manifold.

Let (\mathcal{Z}, g) be a 3-dimensional Riemannian manifold, and $\star 1$ the canonical volume form, $\star : \Gamma \Lambda^q \mathcal{Z} \rightarrow \Gamma \Lambda^{3-q} \mathcal{Z}$, ($q \in \{0, \dots, 3\}$) the Hodge dual map :

$$\star(\alpha \wedge \gamma) = \iota_Y \star \alpha, \quad \star(f \alpha) = f \star \alpha, \quad \star(\alpha + \beta) = \star \alpha + \star \beta,$$

where $Y \in \Gamma T \mathcal{Z}$ is such that $\gamma = g(Y, -)$, for all $\alpha, \beta \in \Gamma \Lambda^q \mathcal{Z}, \gamma \in \Gamma \Lambda^1 \mathcal{Z}, f \in \Gamma \Lambda^0 \mathcal{Z}, (q \in \{0, \dots, 3\})$. In addition, let $\{\sigma^a\}$ be the set of orthogonal co-frames being dual to $\{X_a\}$ so that $g = \delta_{ab} \sigma^a \otimes \sigma^b$, $\sigma^a(X_b) = \delta_{ab}$ and $\star 1 = \sigma^1 \wedge \sigma^2 \wedge \sigma^3$. Then, the contravariant metric tensor field is $g^{-1} = \delta_{ab} X^a \otimes X^b$, and $\star(\sigma^b \wedge \sigma^c) = \epsilon_a^{bc} \sigma^a$, where

$$\epsilon^{abc} = \epsilon_a^{bc} = \begin{cases} +1 & \text{even permutation of } a=1, b=2, c=3 \\ -1 & \text{odd permutation of } a=1, b=2, c=3 \\ 0 & \text{other.} \end{cases}$$

The following will be used.

Lemma 4.1. For any $\alpha, \beta \in \Gamma \Lambda^q \mathcal{Z}$ with $q \in \{0, \dots, 3\}$, it follows that

$$\star \star \alpha = \alpha, \quad \text{and} \quad \alpha \wedge \star \beta = \beta \wedge \star \alpha.$$

Lemma 4.2. For any 1-form $\Delta = \Delta_a \sigma^a$ and 2-form $F = (1/2) F_{ab} \sigma^a \wedge \sigma^b$ with $F_{ab} = -F_{ba}$, it follows that

$$\star(\Delta \wedge F) = g^{-1}(\Delta, \star F) = \delta^{ab} \Delta_a (\star F)_b = \Delta_1 F_{23} + \Delta_2 F_{31} + \Delta_3 F_{12}, \quad (21)$$

where $(\star F)_a$ is such that $\star F = (\star F)_a \sigma^a$.

Some examples of the functional derivative introduced in Definition 2.20 are shown below.

Example 4.1. Let α be a 0-form, and ψ a function of α . Consider the functional

$$\tilde{\psi}_{\mathcal{Z}_0}[\alpha] = \int_{\mathcal{Z}_0} \psi(\alpha) \star 1.$$

Then,

$$\frac{\delta \tilde{\psi}_{\mathcal{Z}_0}}{\delta \alpha} = \frac{\partial \psi}{\partial \alpha}.$$

Example 4.2. Let α be a 1-form written as $\alpha = \alpha_a \sigma^a$. Consider the functional

$$\tilde{\psi}_{\mathcal{Z}_0}[\alpha] = \frac{1}{2} \int_{\mathcal{Z}_0} \alpha \wedge \star \alpha = \int_{\mathcal{Z}_0} \psi \star 1, \quad \text{where} \quad \psi = \frac{1}{2} g^{-1}(\alpha, \alpha) = \frac{1}{2} \delta^{ab} \alpha_a \alpha_b.$$

Then,

$$\frac{\delta \tilde{\psi}_{\mathcal{Z}_0}}{\delta \alpha} = \star \alpha, \quad \text{and} \quad \frac{\partial \psi}{\partial \alpha_a} = \delta^{ab} \alpha_b.$$

Example 4.3. Let β be a 2-form written as $\beta = (1/2) \beta_{ab} \sigma^a \wedge \sigma^b$, and $\star \beta = (\star \beta)_a \sigma^a$. Consider

$$\tilde{\psi}_{\mathcal{Z}_0}[\beta] = \frac{1}{2} \int_{\mathcal{Z}_0} \beta \wedge \star \beta = \int_{\mathcal{Z}_0} \psi \star 1, \quad \text{where} \quad \psi = \frac{1}{2} g^{-1}(\star \beta, \star \beta).$$

Then,

$$\frac{\delta \tilde{\psi}_{\mathcal{Z}_0}}{\delta \beta} = \star \beta, \quad \text{and} \quad \frac{\partial \psi}{\partial \beta^a} = \delta_{ab} \beta^b,$$

where

$$\beta^a := \delta^{ab} (\star \beta)_b, \quad \text{so that} \quad \psi = \frac{1}{2} \delta_{ab} \beta^a \beta^b.$$

4.2 Maxwell fields

The forms $\mathbf{e}, \mathbf{h} \in \Gamma\Lambda^1 \mathcal{Z}$ and $\mathbf{D}, \mathbf{B} \in \Gamma\Lambda^2 \mathcal{Z}$ are used for describing Maxwell's equations. Their physical meanings are given as below :

$$\begin{aligned} \mathbf{e} & : \text{1-form electric field} \\ \mathbf{B} & : \text{2-form magnetic induction field} \\ \mathbf{D} & : \text{2-form displacement field} \\ \mathbf{h} & : \text{1-form magnetic field} \end{aligned}$$

With $\mathbf{e}, \mathbf{h} \in \Gamma\Lambda^1 \mathcal{Z}$ and $\mathbf{D}, \mathbf{B} \in \Gamma\Lambda^2 \mathcal{Z}$, Maxwell's equations without source are written as follows.

Definition 4.1. (*Maxwell's equations without source*): The (3+1)-decomposed Maxwell's equations without external source are

$$\dot{\mathbf{D}} = d\mathbf{h}, \quad \dot{\mathbf{B}} = -d\mathbf{e}, \quad d\mathbf{D} = 0, \quad d\mathbf{B} = 0. \quad (22)$$

Here $\dot{\cdot}$ is derivative with respect to time.

Remark 4.1. The two equations, $d\mathbf{D} = 0$ and $d\mathbf{B} = 0$, can be derived from the other two equations, $\dot{\mathbf{D}} = d\mathbf{h}$ and $\dot{\mathbf{B}} = -d\mathbf{e}$, by applying d with $d^2 = 0$.

To obtain closed equations from Maxwell's equations without source, one needs some relations.

Definition 4.2. (*Constitutive relation*): The following relations

1. a relation connecting \mathbf{e} with \mathbf{D}, \mathbf{B}
2. a relation connecting \mathbf{h} with \mathbf{B}, \mathbf{D}

are referred to as constitutive relations.

There are special constitutive relations with some functions on \mathcal{Z} and on possibly time t . The following are the typical ones.

Definition 4.3. (*Permittivity and permeability*): For Maxwell's equations, if $\mathbf{D} = \varepsilon \star \mathbf{e}$ and $\mathbf{B} = \mu \star \mathbf{h}$ with some positive ε and μ that do not depend on $\mathbf{e}, \mathbf{B}, \mathbf{D}, \mathbf{h}$, then ε is referred to as permittivity, and μ as permeability, respectively.

Remark 4.2. If ε or μ depends on $\mathbf{e}, \mathbf{B}, \mathbf{D}, \mathbf{h}$, then Maxwell's equations are nonlinear. In this paper linear case is only considered.

Remark 4.3. Consider the case where $\dot{\varepsilon} = \dot{\mu} = 0$ on some space on \mathcal{Z} . In this case, one can introduce a potential 1-form \mathbf{A} such that $\mathbf{B} = d\mathbf{A}$. It then follows that $\mathbf{e} = -\dot{\mathbf{A}}$, and one has the equation of motion for \mathbf{A}

$$\ddot{\mathbf{A}} + \frac{1}{\varepsilon} \star d \left(\frac{1}{\mu} \star d\mathbf{A} \right) = 0.$$

From this, one can consider various electromagnetic systems. For example, choose $\varepsilon(\zeta_3) = \varepsilon_0 \operatorname{sech}^2(\zeta_3/\zeta_{30})$ and $\mu = \mu_0$ where ζ_3 is a coordinate for \mathcal{Z} and $\varepsilon_0, \zeta_{30}, \mu_0$ constants. Then one has analytical expressions of electromagnetic waves for this model[30]. In addition choose $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ with ε_0 and μ_0 being constants. Then one has the vacuum system. In this case there exist various solutions whose field lines form knots[31].

Although there are a variety of constitutive relations, this special class of constitutive relations, involving ε and μ as in Definition 4.3, are only considered in this paper since they are typical and mathematically simple.

The forms $\mathbf{e}, \mathbf{B}, \mathbf{D}, \mathbf{h}$ are classified as follows.

Definition 4.4. (Maxwell fields, induction field and field intensity): The forms $\mathbf{e}, \mathbf{B}, \mathbf{D}, \mathbf{h}$ that are used to describe Maxwell's equations are referred to as the Maxwell fields. In addition, the 2-forms \mathbf{D} and \mathbf{B} are referred to as induction fields, and the 1-forms \mathbf{e} and \mathbf{h} are referred to as field intensities.

Remark 4.4. There exists another classification for these forms[21]. The forms \mathbf{e} and \mathbf{B} are untwisted forms, and \mathbf{D} and \mathbf{h} twisted forms.

Given a medium with given boundary, the forms should satisfy appropriate boundary conditions. Taking into account this, one assumes that the solutions to Maxwell's equations always satisfy such boundary conditions in discussions below.

In physics energy plays a role, and energy functionals are used for continuous mechanics. In electromagnetism, the following functionals can be chosen and used in this paper.

Definition 4.5. (Energy and co-energy functionals): Let $\mathbf{e}, \mathbf{B}, \mathbf{D}, \mathbf{h}$ be solutions to Maxwell's equations without source, $\varepsilon : \mathcal{Z} \rightarrow \mathbb{R}$ and $\mu : \mathcal{Z} \rightarrow \mathbb{R}$ permittivity and permeability depending on at a point ζ of \mathcal{Z} and time $t \in \mathbb{T}$, respectively, and $\mathcal{Z}_0 \subseteq \mathcal{Z}$ a subspace of \mathcal{Z} . The functional

$$\tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}[\mathbf{D}, \mathbf{B}] := \frac{1}{2} \int_{\mathcal{Z}_0} \left(\frac{1}{\varepsilon} \mathbf{D} \wedge \star \mathbf{D} + \frac{1}{\mu} \mathbf{B} \wedge \star \mathbf{B} \right), \quad (23)$$

is referred to as the energy functional. In addition,

$$\tilde{\varphi}_{\mathcal{Z}_0}^{\text{EM}}[\mathbf{e}, \mathbf{h}] := \frac{1}{2} \int_{\mathcal{Z}_0} (\varepsilon \mathbf{e} \wedge \star \mathbf{e} + \mu \mathbf{h} \wedge \star \mathbf{h}), \quad (24)$$

is referred to as the co-energy functional.

The functional $\tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}$ in (23) depends on induction fields, and this will lead to the induction oriented formulation of Maxwell's equations. On the other hand, $\tilde{\varphi}_{\mathcal{Z}_0}^{\text{EM}}$ in (24) depends on field intensities, and this will lead to the field intensity oriented formulation of Maxwell's equations.

The decomposed Maxwell's equations are written on a 3-dimensional Riemannian manifold (\mathcal{Z}, g) . With the Riemannian metric tensor field g one can write (23) and (24) as follows.

Lemma 4.3. The functionals (23) and (24) can be written as

$$\tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}[\mathbf{D}, \mathbf{B}] = \int_{\mathcal{Z}_0} \psi^{\text{EM}}(\mathbf{D}, \mathbf{B}) \star 1, \quad \text{and} \quad \tilde{\varphi}_{\mathcal{Z}_0}^{\text{EM}}[\mathbf{e}, \mathbf{h}] = \int_{\mathcal{Z}_0} \varphi^{\text{EM}}(\mathbf{e}, \mathbf{h}) \star 1,$$

where

$$\psi^{\text{EM}}(\mathbf{D}, \mathbf{B}) = \frac{1}{2} \left[\frac{1}{\varepsilon} g^{-1}(\star \mathbf{D}, \star \mathbf{D}) + \frac{1}{\mu} g^{-1}(\star \mathbf{B}, \star \mathbf{B}) \right], \quad \varphi^{\text{EM}}(\mathbf{e}, \mathbf{h}) = \frac{1}{2} [\varepsilon g^{-1}(\mathbf{e}, \mathbf{e}) + \mu g^{-1}(\mathbf{h}, \mathbf{h})]. \quad (25)$$

Proof. With the identities

$$\frac{1}{\varepsilon} \mathbf{D} \wedge \star \mathbf{D} + \frac{1}{\mu} \mathbf{B} \wedge \star \mathbf{B} = \left[\frac{1}{\varepsilon} g^{-1}(\star \mathbf{D}, \star \mathbf{D}) + \frac{1}{\mu} g^{-1}(\star \mathbf{B}, \star \mathbf{B}) \right] \star 1,$$

and

$$\varepsilon \mathbf{e} \wedge \star \mathbf{e} + \mu \mathbf{h} \wedge \star \mathbf{h} = [\varepsilon g^{-1}(\mathbf{e}, \mathbf{e}) + \mu g^{-1}(\mathbf{h}, \mathbf{h})] \star 1,$$

one can complete the proof. \square

Definition 4.6. (Energy density function and co-energy density function): The functions ψ^{EM} and φ^{EM} in (25) are referred to as the energy density function and the co-energy density function, respectively.

In the following, various formulae are shown. They will be used in the rest of this section and Section 5.

Lemma 4.4.

$$\frac{\delta \tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}}{\delta \mathbf{D}} = \frac{1}{\varepsilon} \star \mathbf{D}, \quad \frac{\partial \psi^{\text{EM}}}{\partial D^a} = \frac{1}{\varepsilon} (\star \mathbf{D})_a = \frac{1}{\varepsilon} \delta_{ab} D^b = \left(\frac{\delta \tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}}{\delta \mathbf{D}} \right)_a, \quad (26)$$

where $D^a := \delta^{ab} (\star \mathbf{D})_b$,

$$\star \mathbf{D} = (\star \mathbf{D})_a \sigma^a, \quad \text{and} \quad \frac{\delta \tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}}{\delta \mathbf{D}} = \left(\frac{\delta \tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}}{\delta \mathbf{D}} \right)_a \sigma^a.$$

Proof. With Example 4.3, one can easily prove the first equation. In the following, the second equation is proven. One can write \mathbf{D} with $D_{ab} = -D_{ba}$ as

$$\mathbf{D} = \frac{1}{2} D_{bc} \sigma^b \wedge \sigma^c.$$

It is straightforward to show that

$$\star \mathbf{D} = \frac{1}{2} D_{bc} \star (\sigma^b \wedge \sigma^c) = \frac{1}{2} \epsilon_a{}^{bc} D_{bc} \sigma^a,$$

and

$$(\star \mathbf{D})_a = (\star \mathbf{D})(X_a) = \frac{1}{2} \epsilon_a{}^{bc} D_{bc}, \quad \text{and} \quad D^a = \delta^{ab} (\star \mathbf{D})_b = \frac{\epsilon^{abc}}{2} D_{bc}.$$

Then, it follows from

$$g^{-1}(\star \mathbf{D}, \star \mathbf{D}) = \delta^{aa'} \left(\frac{1}{2} \epsilon_a{}^{bc} D_{bc} \right) \left(\frac{1}{2} \epsilon_{a'}{}^{b'c'} D_{b'c'} \right) = \delta_{aa'} \left(\frac{1}{2} \epsilon^{abc} D_{bc} \right) \left(\frac{1}{2} \epsilon^{a'b'c'} D_{b'c'} \right) = \delta_{aa'} D^a D^{a'},$$

that

$$\frac{\partial \psi^{\text{EM}}}{\partial D^a} = \frac{1}{2\varepsilon} \frac{\partial}{\partial D^a} [g^{-1}(\star \mathbf{D}, \star \mathbf{D})] = \frac{1}{2\varepsilon} \frac{\partial}{\partial D^a} [\delta_{bc} D^b D^c] = \frac{1}{\varepsilon} \delta_{ab} D^b = \frac{1}{\varepsilon} (\star \mathbf{D})_a.$$

□

Similar to Lemma 4.4, one has the following.

Lemma 4.5.

$$\frac{\delta \tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}}{\delta \mathbf{B}} = \frac{1}{\mu} \star \mathbf{B}, \quad \frac{\partial \psi^{\text{EM}}}{\partial B^a} = \frac{1}{\mu} (\star \mathbf{B})_a = \frac{1}{\mu} \delta_{ab} B^b = \left(\frac{\delta \tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}}{\delta \mathbf{B}} \right)_a, \quad (27)$$

where $B^a := \delta^{ab} (\star \mathbf{B})_b$,

$$\star \mathbf{B} = (\star \mathbf{B})_a \sigma^a, \quad \text{and} \quad \frac{\delta \tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}}{\delta \mathbf{B}} = \left(\frac{\delta \tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}}{\delta \mathbf{B}} \right)_a \sigma^a.$$

Proof. A way to prove this is analogous to the proof of Lemma 4.4. □

With the formulae derived above, one can characterize the energy density function and the co-energy density function. To this end, one defines convex function for smooth functions.

Definition 4.7. (Strictly convex function): Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a convex domain, and f a function of $\{x^a\}$ on \mathcal{A}_0 . If the matrix

$$\frac{\partial^2 f}{\partial x^a \partial x^b}$$

is strictly positive definite, then the function is referred to as a strictly convex function. In addition, the property that f is strictly convex is referred to as convexity.

From this definition, Lemma 4.4, and Lemma 4.5, one has the following.

Lemma 4.6. (*Convexity for ψ^{EM} and φ^{EM}*): The functions ψ^{EM} and φ^{EM} in (25) are strictly convex functions.

Proof. Define $\{x^a\} = \{D^1, D^2, D^3, B^1, B^2, B^3\}$ and $\{p_a\} = \{e_1, e_2, e_3, h_1, h_2, h_3\}$. It then follows from $\varepsilon^{-1} > 0, \mu^{-1} > 0$ due to Definition 4.3 that

$$\frac{\partial^2 \psi^{\text{EM}}}{\partial x^a \partial x^b} = \text{diag}\{\varepsilon^{-1}, \varepsilon^{-1}, \varepsilon^{-1}, \mu^{-1}, \mu^{-1}, \mu^{-1}\},$$

which is strictly positive definite. Similarly one can prove that for φ^{EM} . \square

The following states a relation between the energy density function and the co-energy density function.

Lemma 4.7. (*Total Legendre transform of ψ^{EM}*): The functions ψ^{EM} and φ^{EM} defined in (25) are related with the total Legendre transform :

$$\mathfrak{L}[\psi^{\text{EM}}](\mathbf{e}, \mathbf{h}) = \varphi^{\text{EM}}(\mathbf{e}, \mathbf{h}).$$

Proof. The total Legendre transform of $\mathfrak{L}[\psi^{\text{EM}}]$ is calculated to be

$$\begin{aligned} \mathfrak{L}[\psi^{\text{EM}}](\mathbf{e}, \mathbf{h}) &= \sup_{\{D^a\}, \{B^a\}} [D^a e_a + B^a h_a - \psi^{\text{EM}}] = D_0^a e_a + B_0^a h_a - \psi^{\text{EM}}(D_0, B_0) \\ &= \frac{1}{2\varepsilon} \delta^{ab} e_a e_b + \frac{1}{2\mu} \delta^{ab} h_a h_b = \varphi^{\text{EM}}(\mathbf{e}, \mathbf{h}), \end{aligned}$$

where we have used $D_0 = \{D_0^a\}$ and $B_0 = \{B_0^a\}$ that are the unique solutions to

$$e_a = \frac{\partial \psi^{\text{EM}}}{\partial D^a}(D_0, B_0), \quad h_a = \frac{\partial \psi^{\text{EM}}}{\partial B^a}(D_0, B_0).$$

The uniqueness follows from Lemma 4.6. \square

The following states a relation between the two functionals.

Lemma 4.8. (*Total Legendre transform of $\tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}$*): The total Legendre transform of the functional $\tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}$ in (23) is $\tilde{\varphi}_{\mathcal{Z}_0}^{\text{EM}}$ in (24) :

$$\tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}*}[\mathbf{e}, \mathbf{h}] = \tilde{\varphi}_{\mathcal{Z}_0}^{\text{EM}}[\mathbf{e}, \mathbf{h}].$$

Proof. From Definition 3.7, one has

$$\tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}*} = \int_{\mathcal{Z}_0} \mathfrak{L}[\psi^{\text{EM}}] \star 1.$$

Applying Lemma 4.7, one has

$$\tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}*}[\mathbf{e}, \mathbf{h}] = \int_{\mathcal{Z}_0} \mathfrak{L}[\psi^{\text{EM}}] \star 1 = \int_{\mathcal{Z}_0} \varphi^{\text{EM}}(\mathbf{e}, \mathbf{h}) \star 1 = \tilde{\varphi}_{\mathcal{Z}_0}^{\text{EM}}[\mathbf{e}, \mathbf{h}].$$

\square

5 Contact formulation of Maxwell's equations without source

In this section, from the given energy functional (23) and given co-energy functional (24), Maxwell's equations are formulated.

To this end, the field components of $\mathbf{D}, \mathbf{B}, \mathbf{e}, \mathbf{h}$, and the energy density function, $(D^a, B^a, e_a, h_a, \mathcal{E}), (a \in \{1, 2, 3\})$, are identified with vertical 0-forms that are defined as follows.

Definition 5.1. (Maxwell fields and canonical coordinates of contact manifold): Let $\mathbf{e}, \mathbf{D}, \mathbf{B}, \mathbf{h}$ be the Maxwell fields on a Riemannian manifold (\mathcal{Z}, g) . Then D^a, B^a, e_a, h_a are defined such that

$$D^a := \delta^{ab}(\star \mathbf{D})_b, \quad B^a := \delta^{ab}(\star \mathbf{B})_b, \quad \text{and} \quad \mathbf{e} = e_a \boldsymbol{\sigma}^a, \quad \mathbf{h} = h_a \boldsymbol{\sigma}^a, \quad a \in \{1, 2, 3\}.$$

Definition 5.2. (Contact manifold over a base space for Maxwell's equations): Let (\mathcal{Z}, g) be a 3-dimensional Riemannian manifold, $(\mathcal{K}, \pi, \mathcal{Z})$ a bundle over \mathcal{Z} and the fiber space $\pi^{-1}(\zeta)$ equals to a 13-dimensional manifold \mathcal{C}_ζ with $\mathcal{K} = \bigcup_{\zeta \in \mathcal{Z}} \mathcal{C}_\zeta$, $(x^{\text{EM}}, p^{\text{EM}}, z^{\text{EM}})$ canonical coordinates for the fiber space which are

$$x^{\text{EM}} = \{D^1, D^2, D^3, B^1, B^2, B^3\}, \quad p^{\text{EM}} = \{e_1, e_2, e_3, h_1, h_2, h_3\}, \quad z^{\text{EM}} = \mathcal{E},$$

with \mathcal{E} being either an energy density function or a co-energy density function, and λ_V the following contact vertical form

$$\lambda_V = d_V z - p_a d_V x^a.$$

The contact manifold over the base space \mathcal{Z} for the decomposed Maxwell's equations without source in media, (22), is $(\mathcal{K}, \lambda_V, \pi, \mathcal{Z})$.

To close the Maxwell's equations, the constitutive relations

$$e_a = \frac{\partial \psi^{\text{EM}}}{\partial D^a}, \quad h_a = \frac{\partial \psi^{\text{EM}}}{\partial B^a}, \quad a \in \{1, 2, 3\}, \quad (28)$$

or

$$D^a = \frac{\partial \varphi^{\text{EM}}}{\partial e_a}, \quad B^a = \frac{\partial \varphi^{\text{EM}}}{\partial h_a}, \quad a \in \{1, 2, 3\}, \quad (29)$$

are imposed. In addition, one specifies the electromagnetic energy as either

$$\mathcal{E} = \psi^{\text{EM}} \quad \text{or} \quad \mathcal{E} = D^a e_a + B^a h_a - \varphi^{\text{EM}}.$$

From a viewpoint of differential geometry, these relations are conditions that a solution space of Maxwell's equations is a Legendre submanifold generated by ψ^{EM} and that by φ^{EM} .

The fiber space is a 13-dimensional space that is for the expressing unrestricted fields and energy $(D^a, B^a, e_a, h_a, \mathcal{E})$, and the Legendre submanifold is for expressing the restricted fields and energy.

5.1 D - B oriented formulation

In this subsection from the given energy functional (23), Maxwell's equations will be formulated.

Impose $\mathcal{E} = \psi^{\text{EM}}$ and the constitutive relations (28)

$$\mathbf{e} = \frac{\delta \tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}}{\delta \mathbf{D}}, \quad \mathbf{h} = \frac{\delta \tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}}{\delta \mathbf{B}},$$

or equivalently

$$\mathbf{e} = \frac{1}{\varepsilon} \star \mathbf{D}, \quad \mathbf{h} = \frac{1}{\mu} \star \mathbf{B}, \quad \text{or} \quad e_a = \frac{1}{\varepsilon} \delta_{ab} D^b, \quad h_a = \frac{1}{\mu} \delta_{ab} B^b,$$

so that Maxwell's equations are closed ones.

In this contact geometric formulation, Maxwell's equations are realized on the Legendre submanifold of the vertical space generated by ψ^{EM} . Physically this submanifold is the subspace where the energy is properly chosen and constitutive relations (28) are satisfied. To describe this, one introduces the following adapted functions on the fiber space.

Definition 5.3. (Adapted functions for D - B oriented formulation):

$$\Delta_0^{\zeta \psi^{\text{EM}}} := \psi^{\text{EM}} - \mathcal{E}, \quad \Delta_a^{\zeta \psi^{\text{EM}}} := \frac{\partial \psi^{\text{EM}}}{\partial D^a} - e_a, \quad \Delta_{a+3}^{\zeta \psi^{\text{EM}}} := \frac{\partial \psi^{\text{EM}}}{\partial B^a} - h_a, \quad a \in \{1, 2, 3\}.$$

Associated with this set of functions, the following are introduced.

Definition 5.4. (*Adapted mixed forms for D - B oriented formulation*): Let $\Delta_0^{\zeta\psi^{\text{EM}}} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{0,0}\mathcal{K}$, $\Delta_{De}^{\zeta\psi^{\text{EM}}} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{1,0}\mathcal{K}$ and $\Delta_{Bh}^{\zeta\psi^{\text{EM}}} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{1,0}\mathcal{K}$ be such that

$$\Delta_{\mathcal{E}}^{\zeta\psi^{\text{EM}}} := \Delta_0^{\zeta\psi^{\text{EM}}}, \quad \Delta_{De}^{\zeta\psi^{\text{EM}}} := \frac{\delta\tilde{\psi}_{Z_0}^{\text{EM}}}{\delta D} - e, \quad \Delta_{Bh}^{\zeta\psi^{\text{EM}}} := \frac{\delta\tilde{\psi}_{Z_0}^{\text{EM}}}{\delta B} - h.$$

With the adapted functions, Maxwell's equations are formulated in the following space.

Definition 5.5. (*Phase space for the D - B formulation of Maxwell's equations*): Let $\mathcal{A}_{\zeta\psi^{\text{EM}}}^{\mathcal{C}}$ be the Legendre submanifold of the vertical space generated by ψ^{EM} as

$$\mathcal{A}_{\zeta\psi^{\text{EM}}}^{\mathcal{C}} = \left\{ (x^{\text{EM}}, p^{\text{EM}}, z^{\text{EM}}) \in \pi^{-1}(\zeta) \mid \Delta_0^{\zeta\psi^{\text{EM}}} = \Delta_1^{\zeta\psi^{\text{EM}}} = \dots = \Delta_6^{\zeta\psi^{\text{EM}}} = 0 \right\}.$$

Then the sub-bundle $(\mathcal{A}_{\psi^{\text{EM}}}^{\mathcal{K}}, \pi|_{\mathcal{A}_{\psi^{\text{EM}}}^{\mathcal{K}}}, \mathcal{Z})$ with $\mathcal{A}_{\psi^{\text{EM}}}^{\mathcal{K}} := \bigcup_{\zeta \in \mathcal{Z}} \mathcal{A}_{\zeta\psi^{\text{EM}}}^{\mathcal{C}}$ is referred to as the phase space for the D - B formulation of Maxwell's equations (see (15)).

This phase space can also be written as the adapted mixed forms as follows.

Lemma 5.1.

$$\left\{ \Delta_0^{\zeta\psi^{\text{EM}}} = \Delta_1^{\zeta\psi^{\text{EM}}} = \dots = \Delta_6^{\zeta\psi^{\text{EM}}} = 0 \right\} = \left\{ \Delta_{\mathcal{E}}^{\zeta\psi^{\text{EM}}} = \Delta_{De}^{\zeta\psi^{\text{EM}}} = \Delta_{Bh}^{\zeta\psi^{\text{EM}}} = 0 \right\}.$$

Proof. It can be proven with Lemma 4.4 and 4.5. \square

Then the following is one of the main theorems in this paper. On the phase space for the D - B formulation of Maxwell's equations, one has the Maxwell's equations.

Theorem 5.1. (*Maxwell's equation without source in media, induction oriented formulation*): Choose the contact Hamiltonian functional as

$$\tilde{h}_{\psi^{\text{EM}}} = \int_{Z_0} h_{\psi^{\text{EM}}} \star 1 = \int_{Z_0} \left[\Delta_{De}^{\zeta\psi^{\text{EM}}} \wedge F_{\psi^{\text{EM}}}^{\zeta De} + \Delta_{Bh}^{\zeta\psi^{\text{EM}}} \wedge F_{\psi^{\text{EM}}}^{\zeta Bh} + \Gamma_{\zeta\psi^{\text{EM}}} \left(\Delta_{\mathcal{E}}^{\zeta\psi^{\text{EM}}} \right) \star 1 \right],$$

where $h_{\psi^{\text{EM}}} \in \Gamma\Lambda_{\mathbb{V}}^{0,0}\mathcal{K}$, $F_{\psi^{\text{EM}}}^{\zeta De}, F_{\psi^{\text{EM}}}^{\zeta Bh} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{2,0}\mathcal{K}$ are

$$h_{\psi^{\text{EM}}} = \star \left[\Delta_{De}^{\zeta\psi^{\text{EM}}} \wedge F_{\psi^{\text{EM}}}^{\zeta De} \right] + \star \left[\Delta_{Bh}^{\zeta\psi^{\text{EM}}} \wedge F_{\psi^{\text{EM}}}^{\zeta Bh} \right] + \Gamma_{\zeta\psi^{\text{EM}}} \left(\Delta_0^{\zeta\psi^{\text{EM}}} \right),$$

$$F_{\psi^{\text{EM}}}^{\zeta De} := d \left(\frac{1}{\mu} \star B \right), \quad \text{and} \quad F_{\psi^{\text{EM}}}^{\zeta Bh} := -d \left(\frac{1}{\varepsilon} \star D \right),$$

respectively, and $\Gamma_{\zeta\psi^{\text{EM}}}$ is such that

$$\Gamma_{\zeta\psi^{\text{EM}}} \left(\Delta_{\mathcal{E}}^{\zeta\psi^{\text{EM}}} \right) = \begin{cases} 0 & \text{for } \Delta_{\mathcal{E}}^{\zeta\psi^{\text{EM}}} = 0 \\ \text{non-zero} & \text{for } \Delta_{\mathcal{E}}^{\zeta\psi^{\text{EM}}} \neq 0 \end{cases}.$$

Then the restricted contact Hamiltonian vertical vector field $X_{\tilde{h}_{\psi^{\text{EM}}}}|_{\tilde{h}_{\psi^{\text{EM}}}=0}$ gives Maxwell's equations without source and the Poynting theorem.

Proof. In this proof the relation $\mathcal{A}_{\zeta\psi^{\text{EM}}}^{\mathcal{C}} = \{\tilde{h}_{\psi^{\text{EM}}} = 0\}$ is used.

To write the component expression of the contact Hamiltonian vertical vector field, one rewrites $h_{\psi \text{ EM}}$. Writing

$$\begin{aligned}\Delta_{De}^{\zeta\psi \text{ EM}} &= \left(\Delta_{De}^{\zeta\psi \text{ EM}} \right)_a \sigma^a, & \star F_{\psi \text{ EM}}^{\zeta De} &= \left(\star F_{\psi \text{ EM}}^{\zeta De} \right)_a \sigma^a, \\ \Delta_{Bh}^{\zeta\psi \text{ EM}} &= \left(\Delta_{Bh}^{\zeta\psi \text{ EM}} \right)_a \sigma^a, & \star F_{\psi \text{ EM}}^{\zeta Bh} &= \left(\star F_{\psi \text{ EM}}^{\zeta Bh} \right)_a \sigma^a,\end{aligned}$$

with (21), one has that

$$\begin{aligned}h_{\psi \text{ EM}} &= g^{-1} \left(\Delta_{De}^{\zeta\psi \text{ EM}}, \star F_{\psi \text{ EM}}^{\zeta De} \right) + g^{-1} \left(\Delta_{Bh}^{\zeta\psi \text{ EM}}, \star F_{\psi \text{ EM}}^{\zeta Bh} \right) + \Gamma_{\zeta\psi \text{ EM}} \left(\Delta_{\mathcal{E}}^{\zeta\psi \text{ EM}} \right) \\ &= \delta^{ab} \left(\Delta_{De}^{\zeta\psi \text{ EM}} \right)_a \left(\star F_{\psi \text{ EM}}^{\zeta De} \right)_b + \delta^{ab} \left(\Delta_{Bh}^{\zeta\psi \text{ EM}} \right)_a \left(\star F_{\psi \text{ EM}}^{\zeta Bh} \right)_b + \Gamma_{\zeta\psi \text{ EM}} \left(\Delta_{\mathcal{E}}^{\zeta\psi \text{ EM}} \right).\end{aligned}$$

In addition, it follows from (26) and (27) that

$$\left(\Delta_{De}^{\zeta\psi \text{ EM}} \right)_a = \frac{1}{\varepsilon} \delta_{ab} D^b - e_a, \quad \text{and} \quad \left(\Delta_{Bh}^{\zeta\psi \text{ EM}} \right)_a = \frac{1}{\mu} \delta_{ab} B^b - h_a.$$

Then the component expression of the restricted contact vertical vector field is obtained from (12) as

$$\begin{aligned}\dot{D}^a \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c} &= - \frac{\partial h_{\psi \text{ EM}}}{\partial e_a} \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c} = \delta^{ab} \left(\star F_{\psi \text{ EM}}^{\zeta De} \right)_b \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c}, \\ \dot{B}^a \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c} &= - \frac{\partial h_{\psi \text{ EM}}}{\partial h_a} \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c} = \delta^{ab} \left(\star F_{\psi \text{ EM}}^{\zeta Bh} \right)_b \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c}, \\ \dot{e}_a \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c} &= \left(\frac{\partial h_{\psi \text{ EM}}}{\partial D^a} + e_a \frac{\partial h_{\psi \text{ EM}}}{\partial \mathcal{E}} \right) \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c} = \frac{1}{\varepsilon} \left(\star F_{\psi \text{ EM}}^{\zeta De} \right)_a \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c}, \\ \dot{h}_a \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c} &= \left(\frac{\partial h_{\psi \text{ EM}}}{\partial B^a} + h_a \frac{\partial h_{\psi \text{ EM}}}{\partial \mathcal{E}} \right) \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c} = \frac{1}{\mu} \left(\star F_{\psi \text{ EM}}^{\zeta Bh} \right)_a \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c}, \\ \dot{\mathcal{E}} \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c} &= \left(h_{\psi \text{ EM}} - e_a \frac{\partial h_{\psi \text{ EM}}}{\partial e_a} - h_a \frac{\partial h_{\psi \text{ EM}}}{\partial h_a} \right) \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c} \\ &= \left[g^{-1} \left(e, \star F_{\psi \text{ EM}}^{\zeta De} \right) + g^{-1} \left(h, \star F_{\psi \text{ EM}}^{\zeta Bh} \right) \right] \Big|_{\mathcal{A}_{\zeta\psi \text{ EM}}^c}.\end{aligned}$$

These are equivalent to write

$$\dot{D} = d\mathbf{h}, \quad \dot{B} = -de, \quad \dot{e} = \frac{1}{\varepsilon} \star d\mathbf{h}, \quad \dot{h} = -\frac{1}{\mu} \star de, \quad \text{on } \mathcal{A}_{\zeta\psi \text{ EM}}^c,$$

and

$$\dot{\mathcal{E}} = \dot{\psi}^{\text{EM}} = g^{-1}(e, \star d\mathbf{h}) - g^{-1}(h, \star de) = \star(e \wedge d\mathbf{h} - h \wedge de) = -\star d(e \wedge h) \quad \text{on } \mathcal{A}_{\zeta\psi \text{ EM}}^c.$$

The last equation above yields the Poynting theorem :

$$\frac{d}{dt} \tilde{\psi}_{\mathcal{Z}_0}^{\text{EM}}[\mathbf{D}, \mathbf{B}] = \int_{\mathcal{Z}_0} \dot{\psi}^{\text{EM}}(\mathbf{D}, \mathbf{B}) \star 1 = - \int_{\mathcal{Z}_0} d(e \wedge h) = - \int_{\partial \mathcal{Z}_0} e \wedge h,$$

where $\partial \mathcal{Z}_0$ is the boundary of \mathcal{Z} , and Stokes' formula has been used for the last equality.

So far the discussion above is carried out on $\mathcal{A}_{\zeta\psi}^c$ and that is valid for an open covering U_i containing ζ . Taking into account this, one completes the proof. \square

Remark 5.1. The equation involving the Poynting 2-form $e \wedge h$ expresses the energy-balance for the system.

5.2 e - h oriented formulation

In this subsection from the given co-energy functional (24), Maxwell's equations will be formulated.

Impose $\mathcal{E} = D^a e_a + B^a h_a - \varphi^{\text{EM}}$ and the constitutive relations (29)

$$D = \frac{\delta \tilde{\varphi}_{\mathcal{Z}_0^{\text{EM}}}}{\delta e}, \quad B = \frac{\delta \tilde{\varphi}_{\mathcal{Z}_0^{\text{EM}}}}{\delta h},$$

or equivalently,

$$D = \varepsilon \star e, \quad B = \mu \star h, \quad \text{or} \quad D^a = \varepsilon \delta^{ab} e_b, \quad B^a = \mu \delta^{ab} h_b,$$

so that Maxwell's equations are closed ones.

Similar to Definition 5.3, one defines the following.

Definition 5.6. (*Adapted functions for e - h oriented formulation*):

$$\Delta_{\zeta\varphi^{\text{EM}}}^0 := D^a e_a + B^a h_a - \varphi^{\text{EM}} - \mathcal{E}, \quad \Delta_{\zeta\varphi^{\text{EM}}}^a := D^a - \frac{\partial \varphi^{\text{EM}}}{\partial e_a}, \quad \Delta_{\zeta\varphi^{\text{EM}}}^{a+3} := B^a - \frac{\partial \varphi_{\mathcal{Z}_0^{\text{EM}}}^{\text{EM}}}{\partial h_a}, \quad a \in \{1, 2, 3\}.$$

Associated with this set of definitions, the following are introduced.

Definition 5.7. (*Adapted mixed forms for e - h oriented formulation*): Let $\Delta_{\zeta\varphi^{\text{EM}}}^0 \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{0,0}\mathcal{K}$, $\Delta_{\zeta\varphi^{\text{EM}}}^{De} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{2,0}\mathcal{K}$ and $\Delta_{\zeta\varphi^{\text{EM}}}^{Bh} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{2,0}\mathcal{K}$ be such that

$$\Delta_{\zeta\varphi^{\text{EM}}}^{\mathcal{E}} := \Delta_{\zeta\varphi^{\text{EM}}}^0, \quad \Delta_{\zeta\varphi^{\text{EM}}}^{De} := D - \frac{\delta \tilde{\varphi}_{\mathcal{Z}_0^{\text{EM}}}^{\text{EM}}}{\delta e}, \quad \Delta_{\zeta\varphi^{\text{EM}}}^{Bh} := B - \frac{\delta \tilde{\varphi}_{\mathcal{Z}_0^{\text{EM}}}^{\text{EM}}}{\delta h}.$$

With the adapted functions, Maxwell's equations are formulated in the following space.

Definition 5.8. (*Phase space for the e - h formulation of Maxwell's equations*): Let $\mathcal{A}_{\zeta\varphi^{\text{EM}}}^{\mathcal{C}}$ be the Legendre submanifold of the vertical space generated by φ^{EM} as

$$\mathcal{A}_{\zeta\varphi^{\text{EM}}}^{\mathcal{C}} = \left\{ (x^{\text{EM}}, p^{\text{EM}}, z^{\text{EM}}) \in \pi^{-1}(\zeta) \mid \Delta_{\zeta\varphi^{\text{EM}}}^0 = \Delta_{\zeta\varphi^{\text{EM}}}^1 = \dots = \Delta_{\zeta\varphi^{\text{EM}}}^6 = 0 \right\}.$$

Then the sub-bundle $(\mathcal{A}_{\varphi^{\text{EM}}}^{\mathcal{K}}, \pi|_{\mathcal{A}_{\varphi^{\text{EM}}}^{\mathcal{K}}}, \mathcal{Z})$ with $\mathcal{A}_{\varphi^{\text{EM}}}^{\mathcal{K}} := \bigcup_{\zeta \in \mathcal{Z}} \mathcal{A}_{\zeta\varphi^{\text{EM}}}^{\mathcal{C}}$ is referred to as the phase space for the e - h formulation of Maxwell's equations (see (16)).

This phase space can also be written as the adapted mixed forms as follows.

Lemma 5.2.

$$\left\{ \Delta_{\zeta\varphi^{\text{EM}}}^0 = \Delta_{\zeta\varphi^{\text{EM}}}^1 = \dots = \Delta_{\zeta\varphi^{\text{EM}}}^6 = 0 \right\} = \left\{ \Delta_{\zeta\varphi^{\text{EM}}}^{\mathcal{E}} = \Delta_{\zeta\varphi^{\text{EM}}}^{De} = \Delta_{\zeta\varphi^{\text{EM}}}^{Bh} = 0 \right\}.$$

Proof. It can be proven with Lemma 4.4 and 4.5. □

There exists a relation between the phase space for the D - B formulation of Maxwell's equations and that for e - h one.

Proposition 5.1. (*Relation between $\mathcal{A}_{\psi^{\text{EM}}}^{\mathcal{C}}$ and $\mathcal{A}_{\zeta\varphi^{\text{EM}}}^{\mathcal{C}}$*): The subspace $\mathcal{A}_{\psi^{\text{EM}}}^{\mathcal{C}}$ in Definition 5.5 is diffeomorphic to $\mathcal{A}_{\zeta\varphi^{\text{EM}}}^{\mathcal{C}}$ in Definition 5.8, (see also Remark 2.3).

Proof. One can prove this by observing that φ^{EM} is the total Legendre transform of ψ^{EM} due to Lemma 4.7. □

Then the following is the counterpart of Theorem 5.1, and one of the main theorems in this paper. On the phase space for the e - h formulation of Maxwell's equations, one has the Maxwell's equations.

Theorem 5.2. (Maxwell's equation without source in media, field intensity oriented formulation):
Choose the contact Hamiltonian functional as

$$\tilde{h}_{\varphi^{\text{EM}}} = \int_{\mathcal{Z}_0} h_{\varphi^{\text{EM}}} \star 1 = \int_{\mathcal{Z}_0} \left[\Delta_{\zeta\varphi^{\text{EM}}}^{D\mathbf{e}} \wedge \mathbf{F}_{\zeta D\mathbf{e}}^{\varphi^{\text{EM}}} + \Delta_{\zeta\varphi^{\text{EM}}}^{B\mathbf{h}} \wedge \mathbf{F}_{\zeta B\mathbf{h}}^{\varphi^{\text{EM}}} + \Gamma^{\zeta\varphi^{\text{EM}}} \left(\Delta_{\zeta\varphi^{\text{EM}}}^{\varepsilon} \right) \star 1 \right],$$

where $h_{\varphi^{\text{EM}}} \in \Gamma\Lambda_{\mathbb{V}}^0\mathcal{K}$, $\mathbf{F}_{\zeta D\mathbf{e}}^{\varphi^{\text{EM}}}, \mathbf{F}_{\zeta B\mathbf{h}}^{\varphi^{\text{EM}}} \in \Gamma\Lambda_{\mathbb{H},\mathbb{V}}^{1,0}\mathcal{K}$ are

$$h_{\varphi^{\text{EM}}} = \star \left[\Delta_{\zeta\varphi^{\text{EM}}}^{D\mathbf{e}} \wedge \mathbf{F}_{\zeta D\mathbf{e}}^{\varphi^{\text{EM}}} \right] + \star \left[\Delta_{\zeta\varphi^{\text{EM}}}^{B\mathbf{h}} \wedge \mathbf{F}_{\zeta B\mathbf{h}}^{\varphi^{\text{EM}}} \right] + \Gamma^{\zeta\varphi^{\text{EM}}} \left(\Delta_{\zeta\varphi^{\text{EM}}}^{\varepsilon} \right),$$

$$\mathbf{F}_{\zeta D\mathbf{e}}^{\varphi^{\text{EM}}} := \frac{1}{\varepsilon} \star d\mathbf{h}, \quad \text{and} \quad \mathbf{F}_{\zeta B\mathbf{h}}^{\varphi^{\text{EM}}} := -\frac{1}{\mu} \star d\mathbf{e},$$

respectively, and $\Gamma^{\zeta\varphi^{\text{EM}}}$ is such that

$$\Gamma^{\zeta\varphi^{\text{EM}}} \left(\Delta_{\zeta\varphi^{\text{EM}}}^{\varepsilon} \right) = \begin{cases} 0 & \text{for } \Delta_{\zeta\varphi^{\text{EM}}}^{\varepsilon} = 0 \\ \text{non-zero} & \text{for } \Delta_{\zeta\varphi^{\text{EM}}}^{\varepsilon} \neq 0 \end{cases}.$$

Then the restricted contact Hamiltonian vertical vector field $X_{\tilde{h}_{\varphi^{\text{EM}}}}|_{\tilde{h}_{\varphi^{\text{EM}}}=0}$ gives Maxwell's equations without source and the Poynting theorem.

Proof. A way to prove this is analogous to the proof of Theorem 5.1. \square

6 Information geometry for Maxwell's equations

It has been shown in Ref. [7] that a contact manifold and a strictly convex function induce a dually flat space that is used in information geometry.

Since the energy density function and co-energy density function are of strictly convex functions due to Lemma 4.6, one can introduce a dually flat space on a fiber space of a bundle for the Maxwell fields. First, one introduces a metric tensor field as follows.

Definition 6.1. (Fiber metric tensor field for the Maxwell fields): Let ψ^{EM} be an energy density function defined in (25). Then the metric tensor field $g^{\zeta\text{EM}} = g_{ab}^{\zeta\text{EM}} d_{\mathbb{V}} x^a \otimes d_{\mathbb{V}} x^b$ on $\mathcal{A}_{\zeta\psi^{\text{EM}}}^{\mathcal{C}} (\subset \pi^{-1}(\zeta), \zeta \in \mathcal{B})$ with

$$g_{ab}^{\zeta\text{EM}} = \frac{\partial^2 \psi^{\text{EM}}}{\partial x^a \partial x^b}, \quad a, b \in \{1, \dots, 6\} \quad (30)$$

and $\{x^a\} = x^{\text{EM}} = \{D^1, D^2, D^3, B^1, B^2, B^3\}$ is referred to as the fiber metric tensor field of $\mathcal{A}_{\zeta\psi^{\text{EM}}}^{\mathcal{C}}$ for the Maxwell fields.

Noticing Lemma 4.7, one can show the following.

Proposition 6.1. (Components of the contravariant metric tensor field for the Maxwell fields): The inverse matrix of $\{g_{ab}^{\zeta\text{EM}}\}$ in (30) is given as

$$g_{\zeta\text{EM}}^{ab} = \frac{\partial^2 \varphi^{\text{EM}}}{\partial p_a \partial p_b}, \quad a, b \in \{1, \dots, 6\},$$

where $\{p_a\} = p^{\text{EM}} = \{e_1, e_2, e_3, h_1, h_2, h_3\}$.

Proof. A proof is similar to that found in Ref. [16]. \square

In the standard information geometry there are two special coordinates, and analogous coordinates exist for our formulation of Maxwell's equations.

Proposition 6.2. (Dual coordinates for the Maxwell fields): With $x^j = \partial \varphi^{\text{EM}} / \partial p_j$, one has

$$g^{\zeta \text{EM}} \left(\frac{\partial}{\partial x^b}, \frac{\partial}{\partial p_a} \right) = \delta_b^a,$$

where $\{\partial/\partial x^a\}, \{\partial/\partial p_a\}$ are vertical vectors fields.

Proof. It follows from

$$\frac{\partial x^j}{\partial p_a} = \frac{\partial^2 \varphi^{\text{EM}}}{\partial p_a \partial p_j} = g_{\zeta \text{EM}}^{aj}$$

that

$$g^{\zeta \text{EM}} \left(\frac{\partial}{\partial x^b}, \frac{\partial}{\partial p_a} \right) = g_{ij}^{\zeta \text{EM}} \delta_b^i \frac{\partial x^j}{\partial p_a} = g_{ij}^{\zeta \text{EM}} \delta_b^i g_{\zeta \text{EM}}^{aj} = \delta_b^a.$$

□

Remark 6.1. The coordinates x and p satisfying the conditions above are referred to as the dual coordinates in the standard information geometry.

In the standard information geometry, the dual connections are often discussed. These also can appear in the present geometry.

Definition 6.2. (Dual connections on contact manifolds over a base space): Let ∇^ζ be a connection on the Riemannian manifold $(\mathcal{A}_{\zeta \psi^{\text{EM}}}^c, g^{\zeta \text{EM}})$, and $X_\nabla, Y_\nabla, Z_\nabla$ vertical vector fields. If another connection $\nabla^{\zeta'}$ satisfies

$$X_\nabla [g^{\zeta \text{EM}}(Y_\nabla, Z_\nabla)] = g^{\zeta \text{EM}} (\nabla_{X_\nabla}^\zeta Y_\nabla, Z_\nabla) + g^{\zeta \text{EM}} (Y_\nabla, \nabla_{X_\nabla}^{\zeta'} Z_\nabla),$$

then the two connections ∇^ζ and $\nabla^{\zeta'}$ are referred to as dual connections with respect to $g^{\zeta \text{EM}}$.

A realization of connection components of dual connections have been known. In our present case of ψ^{EM} the following is a trivial identity since ψ^{EM} is a quadratic function.

Proposition 6.3. (Component expression of dual connections in contact manifold over a base space): Defining

$$\Gamma_{abc}^{\zeta(\alpha)} := \frac{1-\alpha}{2} \frac{\partial^3 \psi^{\text{EM}}}{\partial x^a \partial x^b \partial x^c}, \quad \alpha \in \mathbb{R},$$

one has

$$\frac{\partial}{\partial x^a} g_{bc}^{\zeta \text{EM}} = \Gamma_{abc}^{\zeta(\alpha)} + \Gamma_{acb}^{\zeta(-\alpha)}, \quad a, b \in \{1, \dots, 6\}.$$

Proof. Substituting (30) into the left hand side of the equation above, one completes the proof. □

Remark 6.2. The dual connections ∇^ζ and $\nabla^{\zeta'}$ with respect to $g^{\zeta \text{EM}}$ are constructed such that

$$\nabla_{\partial/\partial x^a}^\zeta \frac{\partial}{\partial x^b} = \Gamma_{ab}^{\zeta(\alpha)c} \frac{\partial}{\partial x^c}, \quad \nabla_{\partial/\partial x^a}^{\zeta'} \frac{\partial}{\partial x^b} = \Gamma_{ab}^{\zeta(-\alpha)c} \frac{\partial}{\partial x^c},$$

where $\Gamma_{ab}^{\zeta(\alpha)c}$ and $\Gamma_{ab}^{\zeta(-\alpha)c}$ are such that

$$\Gamma_{abc}^{\zeta(\alpha)} = g_{cj}^{\zeta \text{EM}} \Gamma_{ab}^{\zeta(\alpha)j}, \quad \text{and} \quad \Gamma_{abc}^{\zeta(-\alpha)} = g_{cj}^{\zeta \text{EM}} \Gamma_{ab}^{\zeta(-\alpha)j}.$$

With discussions above, one finds the following main theorem in this section.

Theorem 6.1. (Information geometry for Maxwell's equations): Maxwell's equations in media without source induce the quadruplet $(\mathcal{A}_{\zeta, \psi^{\text{EM}}}^c, g^{\zeta \text{EM}}, \nabla^\zeta, \nabla^{\zeta'})$.

On any Riemannian manifold (\mathcal{M}, g) with a connection ∇ , one can find a dual connection ∇' . Then the quadruplet $(\mathcal{M}, g, \nabla, \nabla')$ is referred to as a dually flat space[16]. In accordance with this, one can introduce such a space in the present geometry as follows.

Definition 6.3. (*Dually flat space for Maxwell's equations*): The quadruplet introduced in Theorem 6.1 is referred to as a dually flat space for Maxwell's equations.

The canonical divergence plays a role in information geometry, and that can be defined in the fiber space as follows.

Definition 6.4. (*Canonical divergence on fiber space*): The function $\mathbb{D}^{\zeta \text{ EM}} : \mathcal{A}_{\zeta \psi \text{ EM}}^{\mathcal{C}} \times \mathcal{A}_{\zeta \psi \text{ EM}}^{\mathcal{C}} \rightarrow \mathbb{R}, (\zeta \in \mathcal{Z})$ such that

$$\mathbb{D}^{\zeta \text{ EM}}(\xi \parallel \xi') := \psi^{\text{EM}}(\xi) + \varphi^{\text{EM}}(\xi') - x^a|_{\xi} p_a|_{\xi'}.$$

is referred to as canonical divergence for the Maxwell fields.

The generalized Pythagorean theorem plays a role in the standard information geometry, and an analogous theorem exists in the present geometry.

Theorem 6.2. (*Generalized Pythagorean theorem for the Maxwell fields*): Let ξ', ξ'' , and ξ''' be points of $\mathcal{A}_{\zeta \psi \text{ EM}}^{\mathcal{C}}$, γ^{ζ} the ∇^{ζ} -geodesic connecting ξ''' and ξ'' , and $\gamma^{\zeta'}$ the $\nabla^{\zeta'}$ -geodesic connecting ξ'' and ξ' . If at the intersection ξ'' the curves γ^{ζ} and $\gamma^{\zeta'}$ are orthogonal with respect to $g^{\zeta \text{ EM}}$, then one has that

$$\mathbb{D}^{\zeta \text{ EM}}(\xi''' \parallel \xi') = \mathbb{D}^{\zeta \text{ EM}}(\xi''' \parallel \xi'') + \mathbb{D}^{\zeta \text{ EM}}(\xi'' \parallel \xi').$$

Proof. A proof is similar to that found in Ref. [16]. □

7 Concluding remarks

This paper offers how Maxwell's equations without source in media are formulated with contact geometry. This formulation is based on the theory of fiber bundles, where a fiber space is identified with a contact manifold and a base space 3-dimensional Riemannian manifold expressing physical space. The Legendre submanifolds of the contact manifold over the base space are equivalent to the spaces where constitutive relations and energy relations hold. An important step in this formulation is to recognize that electromagnetic energy functional can be seen as an analogue of a convex function used in convex analysis. From this viewpoint Legendre duality has been focused, and then the induction oriented formulation and field intensity oriented one have been explicitly shown. This viewpoint also has naturally yielded information geometry of the Maxwell fields.

There are numbers of extensions that follow from this work. They are, for example, to develop a geometric theory of Maxwell's equations that can deal with external sources and non-standard constitutive relations, and to apply some theorems in contact topology to Maxwell's equations. We believe that these future works stemmed from this work will develop the theory for the Maxwell fields and its engineering applications.

Acknowledgments

The author would like to thank Ken Umeno (Kyoto University) for supporting my work, also thank Yosuke Nakata (Shinshu University), Minoru Koga (Nagoya University), and Tatsuaki Wada (Ibaraki University) for giving critical comments on this paper.

References

- [1] V.I. Arnold, *Mathematical Methods of Classical Mechanics* (Berlin: Springer), (1976).

- [2] R. Hermann, *Geometry, Systems and Physics* (New York: Dekker), (1973).
- [3] R. Mrugala, Suken kokyuroku, **1142**, 167–181, (2000).
- [4] R. Mrugala, J.D. Nulton, J.C. Schon and P. Salamon, Rep. Math. Phys. **29**, 109–121, (1991).
- [5] D. Eberard, B.M. Maschke, and A.J. Van Der Schaft, Rep. Math. Phys. **60**, 175–198, (2007).
- [6] A. Bravetti and C.S. Lopez-Monsalvo, and F. Nettel, Ann. Phys. **361**, 377–400 (2015).
- [7] S. Goto, J. Math. Phys. **56**, 073301, (2015).
- [8] R. Mrugala, J.D. Nulton, J.C. Schon and P. Salamon, Phys. Rev. A **41**, 3156–3160, (1990).
- [9] J. Jurkowski, Phys. Rev. E **62**, 1790–1798, (2000).
- [10] A. Bravetti and C.S. Lopez-Monsalvo, J. Phys. A **48**, 125206, (2015).
- [11] R. Ghrist, Handbook of Mathematical Fluid Dynamics, (Elsevier, 2007), **4**, Chapter 1, 1–37, (2007).
- [12] T. Ohsawa, Automatica, **55**, 1–5, (2015).
- [13] A. Bravetti, and D. Tapias, J. Phys. A **48**, 245001, (2014).
- [14] D. Eberard, B.M. Maschke, and A.J. Van Der Schaft, Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems, Kyoto, Japan, July 24–28, (2006).
- [15] A. Bravetti, H. Cruz, and D. Tapias, Ann. Phys., **376**, 17–39, (2017).
- [16] S.I. Amari and H. Nagaoka, *Methods of Information Geometry*, Trans. Math. Monogr. Vol. 191 (Providence: American Mathematical Society) (2000).
- [17] A. Fujiwara and S. Shuto, Phys. Lett. A, **374**, 911–916, (2009).
- [18] A. Ohara and S. Amari, Kybernetika, **30**, 369–386, (1994).
- [19] S. Goto, J. Math. Phys., **57**, 102702, (2016).
- [20] M. Nakahara, *Geometry, Topology and Physics*, Institute of Physics Publishing, (1990).
- [21] T. Frankel, *The geometry of Physics, –An Introduction–*, (Second Edition), Cambridge University Press, (2004).
- [22] I.M. Benn and R.W. Tucker, *An Introduction to Spinors and Geometry With Applications in Physics*, Adam Hilger (1988).
- [23] J.E. Marsden, T. Ratiu, and R. Abraham, *Manifolds, Tensor analysis, and Applications*, (second edition) Springer (1988).
- [24] T. Courant, Trans. Amer. Math. Soc., **319**, 631–661, (1990).
- [25] A. van der Schaft and D. Jeltsema. *Port-Hamiltonian Systems Theory: An Introductory Overview*, Foundations and Trends in Systems and Control, **1**, 173–378, (2014).
- [26] A.J. van der Schaft and B.M. Maschke, J. of Geom. and Phys., **42**, 166–194, (2002).
- [27] H. Yoshimura and J.E. Marsden, J. of Geom. and Phys., **57**, 133–156, (2006).
- [28] G. Blankenstein, IEEE Trans. Circuits Syst., **52**, 396–404, (2005).
- [29] M. Dahl, Progress in Electromagnetic Research, PIER **46**, 77–104, (2004).
- [30] S. Goto, R.W. Tucker, and T.J. Walton, [arXiv:1402.6582](https://arxiv.org/abs/1402.6582).
- [31] M. Arrayas, D. Bouwmeester, and J.L. Trueba, Phys. Rep., **667**, 1–61, (2017).